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MULTIPERIODIC SOLUTIONS OF SYSTEMS EQUATIONS WITH VARIOUS DIFFERENTIATION OPERATORS

ABSTRACT
of the thesis for the degree of Doctor of Philosophy (PhD) in the specialty 6D060100 – Mathematics

Structure and scope of the thesis. The thesis consists of an introduction, 3 chapters (the first chapter includes 4 sections and 3 subsections, the second chapter consists of 2 sections and 6 subsections, the third chapter consists of 2 sections), a conclusion, list of references.

The number of illustrations, tables and references. In the thesis used 3 illustrations, 1 table and 98 references.

Keywords. Differentiation operator, hyperbolicity in the narrow sense, reduction to canonical form, multiperiodicity, matricant, method of projectors, method of characteristics.

The actuality of the thesis is due to the need to develop new effective methods for studying initial problems and multiperiodic solutions of systems of equations with various differentiation operators describing oscillatory processes from hydromechanics, electromagnetic phenomena, etc.

The dissertation research is devoted to the study of multiperiodic solutions of systems of the first order partial differential equations, the main parts of which are determined by linear relations

$$D_j x_j = \frac{\partial x_j}{\partial \tau} + \sum_{k=1}^{m}\sum_{i=1}^{n} a_{ji}^{k} \frac{\partial x_j}{\partial t_k}, \; j = \overline{1,n}, \tag{0.1}$$

where $x_j = x_j(\tau, t)$ are unknown functions with variables $\tau \in R = (-\infty, +\infty)$ and $t = (t_1, ..., t_m) \in R \times ... \times R = R^m$; $a_{ji}^k = a_{ji}^k(\tau, t)$ are given functions. Note that for such systems, in general, there is no developed method for solving the initial problem.

Putting $x = (x_1, ..., x_n)$, the operator (0.1) can be represented as a matrix differentiation operator

$$Dx = \frac{\partial x}{\partial \tau} + \sum_{k=1}^{m} A_k(\tau, t) \frac{\partial x}{\partial t_k} \tag{0.2}$$

in the directions of vector fields determined by the equations of characteristics, where $A_k(\tau, t) = \left[a_{ji}^k(\tau, t)\right]$ is square $n \times n$-matrix with discrete $k = \overline{1,m}$.

Then the general view of the system under study can be written in the form of a quasilinear vector equation

$$Dx = B(\tau, t)x + f(\tau, t, x) \tag{0.3}$$

with a given operator D, matrix $B(\tau, t)$, and vector function $f(\tau, t, x)$ arguments $(\tau, t) \in R \times R^m$, $x \in R_\Delta^n = \{x \in R^n : |x| \le \Delta, \;\; \Delta - const > 0\}$.

In this paper, we study the issues of simplifying the operator (0.2), in the sense of reducing it to a canonical form, develop methods for integrating and studying multiperiodic solutions, in particular, when the matrices A_k and B are constant.

If we turn to the history of the problem of integration of equations (0.3) with the operator (0.2), then in the analytical case it was investigated by S.V.Kovalevskaya (1874-1885) by the method of majoration. Thus, the Cauchy-Kovalevskaya theorem on solving the initial problem for equations (0.3) with the operator (0.2) became widely known.

After the appearance of the classical works of L.Euler (1768-1770) and J.Lagrange (1774) on the creation of the calculus of variations, in particular, on the integration of partial differential equations, S.V.Kovalevskaya in 1888 received fundamental results on solving the problem of the rotation of a solid body around a fixed point, which were awarded the prize of the Paris Academy of Sciences. In this regard, attention should be paid to the importance of studying oscillatory solutions of equations with many frequencies, which are reduced to the question of multiperiodic solutions of equations of the form (0.3) with the operator (2).

The foundations of the general theory of systems of partial differential equations of the first order in the smooth case are laid in the works of I.G.Petrovsky. The main way to solve such systems is the Cauchy characteristic method, the essence of which is to reduce the problem to the integration of ordinary differential equations. Lagrange once called the mixing of such a character an *art*.

One of the methods of implementing the method of characteristics is to bring the operator (0.2) with two variables to the canonical form given in the works of I.G.Petrovsky. In his works, equation (0.3) is called *elliptic* if the matrix of coefficients $A_1 = A(\tau,t)$ has no real roots. In the case of not ellipticity of equation (0.3), but reducibility to canonical form, the equation is called *hyperbolic*.

If each of the matrices has real and different eigenvalues, then equation (0.3) is called *hyperbolic in the narrow sense*. The research method of this work is closely related to the methods of the theory of partial differential equations (0.3), when the matrices $A_k(\tau,t)$ of the operator (0.2) in the scalar case (0.1) have the properties

$$a_{ji}^k(\tau,t)=\begin{cases} b_j^k(\tau,t)\neq 0, i=j, \\ 0, i\neq j, \end{cases} \quad b_j^k(\tau,t)=\lambda_{kj}(\tau,t)\neq 0, \quad i,j=\overline{1,n},\ k=\overline{1,m}. \quad (0.4)$$

This case represents the hyperbolicity of equation (0.3) in a narrow sense.

The canonical form D^* of the operator D has the form

$$D^* = \mathrm{diag}\left[D_\pi^*,...,D_\pi^*\right],\ D_\pi^* = \frac{\partial}{\partial\tau} + \sum_{k=1}^{m}\lambda_k(\tau,t)\frac{\partial}{\partial t_k} \quad (0.5)$$

when the condition (0.4) is met that is, all the equations of the system (0.3) after reduction have the same differentiation operator D_π^*. Then the system (0.3) can be called *parabolic*.

In particular, for $\lambda_k = 1,\ k=\overline{1,m}$ from (0.5) we have the operator

$$D_e^* = \frac{\partial}{\partial \tau} + \sum_{k=1}^{m} \frac{\partial}{\partial t_k} \equiv \frac{\partial}{\partial \tau} + \left\langle e, \frac{\partial}{\partial t} \right\rangle, \qquad (0.6)$$

where $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1}, ..., \frac{\partial}{\partial t_m} \right)$ is vector operator; $e = (1,...,1)$; $\langle \cdot, \cdot \rangle$ is the sign of the scalar product.

Multiperiodic solutions of equations with operator (0.6) are investigated in the works of V.H.Kharasakhal. A task of this nature for systems in a special kind of operator D, which is reduced to a linear operator with operator (0.5) in the multiperiodic case, is sufficiently fully investigated in the works of D.U.Umbetzhanov, Zh.A.Sartabanov, A.B.Berzhanov, etc.

In the case (0.4) of operator D for systems of the form (0.3), the issues of general theory and multiperiodic solutions are not considered in this formulation. This explains the relevance of the study. In this dissertation, these studies are developed for equations of the form (0.3) with the operator (0.2) in the narrow hyperbolic case on the basis of newly developed methods.

Speaking about the relevance of the topic of the dissertation research, it is impossible not to mention the fundamental works on the theory of oscillations.

It is known that the theory of periodic, that is, singlefrequency oscillatory solutions of ordinary differential equations was developed in the works of A.M.Lyapunov and A.Poincare.

The development of this theory on the study of multi-frequency periodic oscillations described by both ordinary differential equations and partial differential equations related to the methods of averaging and integral manifolds is covered in the fundamental works of N.M.Krylov, N.N.Bogolyubov, Yu.A.Mitropolsky, A.M.Samoylenko and their followers.

The rapid development of methods of multi-frequency oscillations was due to the creation of the KAM theory by A.N.Kolmogorov, V.I. Arnold and Y. Moser.

Along with these methods, since the second half of the twentieth century, research methods for multi-frequency oscillations have been developing, based on the transition from ordinary differential equations to partial differential equations of the first order and from the study of quasi-periodic solutions of ordinary differential equations to the study of multi-periodic solutions of partial differential equations. Such a transition is realized thanks to Bohr's theorem on the deep connection of continuous quasi-periodic functions of one variable with continuous periodic functions of many variables. This peculiar method originates from the works of V.Kh. Kharasakhal and D.U. Umbetzhanov and their followers Zh.A. Sartabanov, A.B. Berzhanov and others developed it. The main results on multiperiodic solutions of partial differential equations relate to systems of equations with the same differentiation operator when the matrix differentiation operator is parabolic. The question of the development of methods for the study of multiperiodic solutions of systems of partial differential equations of hyperbolic type raised by D.U.Umbetzhanov, but it remained at the initial stage of development. He investigated the question of the existence and uniqueness of a multiperiodic solution of a system when its corresponding linear system forms independent subsystems with parabolic differentiation operators.

The research of this dissertation work on the part of multiperiodic solutions is devoted to the development of integration methods and the study of the existence of multiperiodic solutions of narrow hyperbolic quasi-linear equations with matrix differentiation operators for variables of arbitrary quantity.

The relevance of the study is related to the applications of equations of the form (0.3) with quasilinear matrix differentiation operators (0.2). For example, *the nonlinear system of Euler equations of fluid motion*, *the convective diffusion equation*, *the telegraphic equation*, etc. The dissertation shows that they can be represented as a system (0.3) with the operator (0.2).

It is also known that hyperbolic equations are called wave equations. In many cases of an applied nature, their solutions are associated with periodically traveling waves (for example, the Dalembert formula). Consequently, issues related to the multi-periodicity of solutions for all variables or parts of them are of particular interest.

Thus, the relevance of the dissertation research is justified both in theoretical and applied aspects. For example, the nonlinear system of Euler equations of fluid motion, the convective diffusion equation, the telegraphic equation.

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Thus, we believe that the relevance of the dissertation research is justified both in theoretical and applied aspects.

The aim of the thesis research is to develop a method for simplifying the differentiation operator, a method for integrating and investigating multiperiodic solutions of linear and quasilinear systems with various differentiation operators, as well as a boundary value problem for a linear hyperbolic system in the narrow sense.

The research problems:

1) develop a method for reducing the matrix operator of differentiation in $m+1$ variables to a linear operator with a matrix operator of differentiation in m variables;

2) establish the existence of multiperiodic zeros of a multiperiodic operator with constant coefficients in the narrow hyperbolic case;

3) develop a methodology for integrating linear systems with two different differentiation operators and establish sufficient conditions for the existence of multiperiodic solutions in the noncritical case;

4) on the basis of the developed method to obtain sufficient conditions for the unique solvability of the initial problem for a quasilinear system with two different differentiation operators and the existence of a unique multiperiodic solution of this system;

5) obtain sufficient conditions for the unique solvability of the initial problem for linear equations with a hyperbolic in the narrow sense differentiation operator and

establish sufficient conditions for the existence of a unique multiperiodic solution of these equations;

6) obtain sufficient conditions for the solvability of the boundary value problem for a system of equations with a hyperbolic in the narrow sense differentiation operator;

7) on the basis of the developed methodology, to obtain sufficient conditions for the unique solvability of the initial problem for a quasilinear system with a hyperbolic in the narrow sense differentiation operator and the existence of a unique multiperiodic solution of this system.

The research methods. The well-known methods and results of the theory of partial differential equations, the theory of oscillations and the theory of operators are widely used in the dissertation work. The main method of research and solving the problems considered in the dissertation are the methods of Kharasakhal-Umbetzhanov and the methods of Zh.A.Sartabanov works on their development, known from monographs and publications of the authors. In the process of researching the problems of this dissertation, new methods have been developed: the method of reductive reduction of the operator to the canonical form; the method of block systems; the method of introducing additional variables; the method of projectors.

The objects of research are solutions of linear and quasilinear systems of several independent variables with various differentiation operators and their multiperiodicity.

The scientific novelties:

a) a method has been developed for reducing the matrix operator of differentiation with respect to $m+1$ variables to a linear operator with a matrix operator of differentiation with respect to m variables, based on the transition along the characteristic of one of the variables;

b) the existence of an infinite set of multiperiodic zeros of a multiperiodic operator with constant coefficients in the narrow hyperbolic case is established;

c) an approach to the integration of linear systems with two different differentiation operators has been developed and sufficient conditions for the existence of multiperiodic solutions in the noncritical case have been established;

d) on the basis of the developed method, sufficient conditions for the unique solvability of the initial problem for a quasilinear system with two different differentiation operators and the existence of a unique multiperiodic solution of this system are obtained;

e) a method of introducing additional variables for linear equations with a hyperbolic in a narrow sense differentiation operator has been developed;

f) a method of projectors of transition from one variable to another for a hyperbolic in the narrow sense system has been developed, sufficient conditions for unique solvability of the initial problem for linear equations with a hyperbolic in the narrow sense differentiation operator have been obtained, and sufficient conditions for the existence of a unique multiperiodic solution of these equations have been established;

g) sufficient conditions for the solvability of the boundary value problem for a system of equations with a hyperbolic in the narrow sense differentiation operator are obtained;

h) on the basis of the developed method, sufficient conditions for the unique solvability of the initial problem for a quasilinear system with a hyperbolic in the narrow sense differentiation operator and the existence of a unique multiperiodic solution of this system are obtained.

The results of the thesis which are taken out on defense:

– a method of reducing a matrix operator of differentiation by variables to a linear operator with a matrix operator of differentiation by variables, based on the transition along the characteristic of one of the variables;

– the existence of an infinite set of multiperiodic zeros of a multiperiodic operator with constant coefficients in the narrow hyperbolic case;

– the method of integrating linear systems with two different differentiation operators and sufficient conditions for the existence of multiperiodic solutions in the noncritical case;

– sufficient conditions for the unique solvability of the initial problem for a quasilinear system with two different differentiation operators and the existence of a unique multiperiodic solution of this system;

– a method of introducing additional variables for linear equations with a hyperbolic in the narrow sense differentiation operator;

– the method of projectors of transition from one variable to another for a hyperbolic in the narrow sense system, sufficient conditions for unique solvability of the initial problem for linear equations with a hyperbolic in the narrow sense differentiation operator and sufficient conditions for the existence of a unique multiperiodic solution of these equations;

– sufficient conditions for the solvability of the boundary value problem for a system of equations with a hyperbolic in the narrow sense differentiation operator;

– sufficient conditions for the unique solvability of the initial problem for a quasilinear system with a hyperbolic in the narrow sense differentiation operator and the existence of a unique multiperiodic solution of this system.

The personal contribution of the author. All the results of the thesis are obtained by the author. The participation of co-authors and scientific consultants consists in setting goals and discussing the results.

Approbation of the received results. The main results of the thesis were reported and discussed at the following conferences and seminars:

– VIII International Scientific Conference "Problems of Differential equations, analysis and Algebra". Aktobe, November 1, 2018;

– The traditional April International Mathematical conference in honor of the Day of Science Workers of the Republic of Kazakhstan and the Workshop "Problems of modeling processes in electrical contacts" dedicated to the 80th anniversary of Academician of the National Academy of Sciences of the Republic of Kazakhstan S.N. Kharin. Almaty, April 3-5, 2019;

– International scientific Conference "Theoretical and applied problems of mathematics, mechanics and computer science", dedicated to the 70th anniversary of Ph.D., Professor M.I. Ramazanov. Karaganda, June 12-13, 2019;

– International Conference "Actual Problems of Analysis, Differential Equations and Algebra" (EMJ-2019) dedicated to the 10th anniversary of the issue of the Eurasian Mathematical Journal. Nur-Sultan, October 16-19, 2019;

– The traditional international April Mathematical conference in honor of the Day of Science Workers of the Republic of Kazakhstan. Almaty, April 5-8, 2020;

– The traditional April International Mathematical conference in honor of the Day of Science Workers of the Republic of Kazakhstan, dedicated to the 75th anniversary of Academician of the National Academy of Sciences of the Republic of Kazakhstan T.Sh. Kalmenov. Almaty, April 5-8, 2021;

– VI International Scientific Conference "Non-local boundary value problems and related problems of mathematical biology, computer science and Physics". Nalchik, December 5-9, 2021;

– IX International Scientific Conference "Problems of Differential Equations, Analysis and Algebra". Aktobe, May 24-28, 2022;

– Scientific seminar "The Problems of Applied Mathematics and Computer Science", Department of Mathematics, K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan (seminar leader Doctor of Physical and Mathematical Sciences, professor Zh. Sartabanov).

Publications. On the topic of the dissertation, 13 articles were published, including 1 publication in a ranking scientific journal indexed in the Scopus database, 4 publications in scientific journal included in the list recommended by the Committee for Control in the Sphere of Education and Science (formerly CCSES) of the Ministry of Education and Science of the Republic of Kazakhstan for publication of the main scientific results of scientific activities, 8 publications in the materials of the international scientific conferences.