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METHODS FOR SOLVING OF INITIAL-BOUNDARY VALUE PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS OF FOURTH ORDER

ABSTRACT
of the thesis for the degree of Doctor of Philosophy (PhD) in the specialty 6D060100 – Mathematics

Structure and scope of the thesis. The dissertation work consists of an introduction, three sections (the first section includes 5 subsections, the second and third sections include three subsections each), a conclusion and list of references.

Keywords. Fourth-order partial differential equations, initial boundary value problems, periodic problem, two-point problem, nonlocal multipoint problem, Goursat type problem, system of pseudo-hyperbolic equations, functional parametrization method, algorithm.

Relevance of the topic. Currently, much attention is paid to the problems of mathematical physics associated with the frequent description of the application as a mathematical model in various physical, chemical, biological processes. The theory of these problems, along with its great applied value, is considered a new theory developing in classical mathematical physics. One of the most important classes of such problems are initial-boundary value problems for fourth-order partial differential equations. The dissertation work is devoted to solutions and methods of finding solutions to initial boundary value problems for fourth-order partial differential equations with two independent variables. For three main classes of systems of partial differential equations of the fourth order, boundary value problems of the Goursat type, periodic, two-point and multi-point problems are investigated. The conditions of existence and uniqueness of the classical solution, methods of finding a solution are considered. The conditions for unique solvability of the problems under consideration are given in terms of the coefficients of the equation and boundary matrices. Algorithms for finding a solution have been developed and convergence conditions have been established.

The purpose of the research creation of research methods and solutions of initial-boundary, periodic, non-local problems for fourth-order partial differential equations with two independent variables.

The object of the study is initial boundary value problems for fourth-order partial differential equations with two independent variables.

Research methods. In the dissertation, the methods of solving initial-boundary and non-local problems for fourth-order partial differential equations with two variables are the method of introducing additional functions, the method of successive approximations, the method of functional parameterization and the method of parameterization by D. S. Dzhumabaev.

Scientific novelty and practical value of the work.

In the dissertation work

a) the system of partial differential equations of the fourth order is studied;

b) a unified method for solving initial boundary value problems for a system of fourth-order partial differential equations is applied;

c) the conditions of unique solvability are defined in the terms of the initial data;

d) algorithms for finding solutions are proposed and convergence is proved.

The results of the dissertation are mainly theoretical in nature. The scientific importance of the work is that for partial differential equations of the fourth order, a constructive method of research and problem solving is constructed. The results obtained in this work can be used in solving boundary value problems for high-order partial differential equations, as well as in studying elective courses at mathematical faculties of universities.

The results of the thesis which are taken out on defense:

- Methods for solving initial boundary value problems for systems of fourth-order partial differential equations;
- Boundary value problems (Goursat type, periodic, two-point and multipoint problems) of the three main classes of systems of partial differential equations of the fourth order;
- Conditions for the existence and uniqueness of the classical solution of the problems under consideration;
- Algorithms for finding solutions and the conditions of their convergence.

The personal contribution of the author. All the results of the thesis are obtained by the author. The participation of co-authors and scientific consultants consists in setting goals and discussing the results.

Publications. On the topic of the dissertation, 13 articles were published, including 3 publications in a ranking scientific journal indexed in the Scopus database, 3 publications in scientific journal included in the list recommended by the Committee for Control in the Sphere of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan for publication of the main scientific results of scientific activities, 7 publications in the materials of the international conferences.

The results of the dissertation work were carried out within the framework of the scientific project "Methods for solving initial-boundary value problems for high-order partial differential equations and their applications" (No. AP05131220, 2018-2020) of the grant program for fundamental research in the field of natural sciences of the Science Committee of the Ministry of Education and Science of the Republic Kazakhstan.

Summary of work

In the first section, a system of fourth-order partial differential equations in the domain $\Omega = [0,T] \times [0,\omega]$ is considered

$$\frac{\partial^4 u}{\partial t \partial x^3} = A_1(t,x)\frac{\partial^3 u}{\partial x^3} + A_2(t,x)\frac{\partial^3 u}{\partial t \partial x^2} + A_3(t,x)\frac{\partial^2 u}{\partial x^2} + A_4(t,x)\frac{\partial^2 u}{\partial t \partial x} +$$
$$+A_5(t,x)\frac{\partial u}{\partial x} + A_6(t,x)\frac{\partial u}{\partial t} + A_7(t,x)u + f(t,x), \qquad \text{(I)}$$

where $u(t,x) = col(u_1(t,x),\ u_2(t,x), \dots, u_n(t,x))$ is an unknown vector-function, $A_i(t,x)$ is $n \times n$-matrices, $i = \overline{1,7}$, where $f(t,x)$ is an n-vector–function continuous in the domain Ω.

In the first subsection, for a system of equations (I)

$$u(t,0) = \psi_1(t), \quad t \in [0,T] \qquad (1)$$
$$\left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0} = \psi_2(t), \quad t \in [0,T] \qquad (2)$$
$$\left.\frac{\partial^2 u(t,x)}{\partial x^2}\right|_{x=0} = \psi_3(t), \quad t \in [0,T] \qquad (3)$$
$$u(0,x) = \varphi(x), \ \ x \in [0,\omega] \qquad (4)$$

with conditions (1)-(4), the problem is reduced to a Goursat-type problem and integral relations for a system of second-order hyperbolic equations with the introduction of new functions, reduction of the order of the given problem and transformation of the initial boundary conditions. Taking into account the initially set conditions, an algorithm was created for finding a solution to the Goursat problem for a system of second-order hyperbolic equations using the method of sequential approximations. Using the solution of the Goursat problem for integral relations, the solution of the originally given problem is determined. Theorems on the convergence of approximate solutions and its uniqueness are proved.

In the second subsection, for the system of equations (I)

$$u(t,0) = \psi_1(t), \quad t \in [0,T] \qquad (5)$$
$$\left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0} = \psi_2(t), \quad t \in [0,T] \qquad (6)$$
$$\left.\frac{\partial^2 u(t,x)}{\partial x^2}\right|_{x=0} = \psi_3(t), \quad t \in [0,T] \qquad (7)$$
$$u(0,x) = u(T,x), \ \ x \in [0,\omega] \qquad (8)$$

the existence and uniqueness of the solution of a periodic problem with conditions (5)-(8) are considered.

A double replacement was made. With the first substitution, the given problem is reduced to equivalent periodic problems for a system of second-order hyperbolic equations and integral relations. For the second time, the periodic problem is substituted for a system of partial derivatives of first-order differential equations and integral relations. The solution of the periodic problem for partial derivatives of first-order differential equations is shown, taking into account the initially set conditions. The solution of the periodic problem for partial derivatives of first-order differential equations using conditions is determined by the method of sequential approximations. Through integral relations, with the help of the solutions found, the solution of the originally given problem is also found. Theorems on the convergence of approximate solutions and its uniqueness are established and proved.

In the third subsection, for the system of equations (I)

$$\frac{\partial^3 u(0,\ x)}{\partial x^3} = K(x)\frac{\partial^3 u(T,x)}{\partial x^3} + \varphi(x), \quad x \in [0,\omega] \tag{9}$$

$$u(t,0) = \psi_0(t), \quad t \in [0,T] \tag{10}$$

$$\left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0} = \psi_1(t), \quad t \in [0,T] \tag{11}$$

$$\left.\frac{\partial^2 u(t,x)}{\partial x^2}\right|_{x=0} = \psi_2(t), \quad t \in [0,T] \tag{12}$$

the existence and uniqueness of the solution of the initial periodic problem with conditions (9)-(12) are investigated. In the course of the study, the initial problem is replaced with unknown functions and the transition to the initial boundary value problem for partial derivatives of second-order differential equations. Using the method of sequential approximations, the integral relations of this problem are also found. For partial derivatives of second-order differential equations, the convergence theorems of approximate solutions of the initial boundary value problem and its uniqueness are proved by the parameterization method of D.S.Dzhumabaev. A special case $K(x) = I$ and $\varphi(x) = 0$ is also considered and a theorem on the existence and uniqueness of the solution is given.

The fourth subsection for the system of equations (I)

$$u(t,0) = \psi_1(t), \quad t \in [0,T], \tag{13}$$

$$\left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0} = \psi_2(t), \quad t \in [0,T], \tag{14}$$

$$\left.\frac{\partial^2 u(t,x)}{\partial x^2}\right|_{x=0} = \psi_3(t), \quad t \in [0,T], \tag{15}$$

$$P(x)u(0,x) + S(x)u(T,x) = \varphi(x), \quad x \in [0,\omega], \tag{16}$$

introduces unknown functions for finding a solution to a two-point problem with condition (13)-(16) and is reduced to an equivalent two-point initial boundary value problem for a system of second-order hyperbolic equations and integral relations. After making the second substitution, they proceed to the two-point problem for partial derivatives of first-order differential equations and integral relations. The solution of the two-point problem for partial derivatives of first-order differential equations, taking into account the initially set conditions, is shown explicitly. The solution of the two-point problem for partial derivatives of first-order differential equations using the given conditions is determined by the method of sequential approximations. With the help of the solutions found, the solution of the initially given problem is also determined through integral relations. Theorems on convergence and uniqueness of the solution are established and proved.

In the fifth subsection

$$\sum_{j=0}^{m}\sum_{i=0}^{3} M_{ij}(x)\frac{\partial^i u(t_j,x)}{\partial x^i} = \varphi(x), \quad x \in [0,\omega], \tag{17}$$

$$u(t,0) = \psi_0(t), \qquad \left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0} = \psi_1(t), \qquad \left.\frac{\partial^2 u(t,x)}{\partial x^2}\right|_{x=0} = \psi_2(t), \tag{18}$$

the existence and uniqueness of the solution of a multipoint problem with conditions (17)-(18) for the system of equations (I) is considered. A multipoint nonlocal problem for a system of partial derivatives of fourth-order differential equations with the introduction of an unknown function is reduced to equivalent multipoint problems for partial derivatives of first-order integro-differential equations and integral relations. To write down the solution of this problem in an explicit form, the solution of partial differential equations of the first order is used. To find a solution to the multipoint problem for partial derivatives of integro-differential equations of the first order, the method of sequential approximation is applied. A theorem is given that formulates the convergence of approximate solutions and its uniqueness. To prove that the conditions of the theorem are fulfilled, the generalized Gronwall-Bellman inequality is used.

In the second chapter, a system of fourth-order partial differential equations in the domain $\Omega = [0,T]\times[0,\omega]$ is considered

$$\frac{\partial^4 u}{\partial t^2\partial x^2} = A_1(t,x)\frac{\partial^3 u}{\partial t\partial x^2} + A_2(t,x)\frac{\partial^3 u}{\partial t^2\partial x} + A_3(t,x)\frac{\partial^2 u}{\partial t\partial x} + A_4(t,x)\frac{\partial^2 u}{\partial t^2} + \\ + A_5(t,x)\frac{\partial^2 u}{\partial x^2} + A_6(t,x)\frac{\partial u}{\partial t} + A_7(t,x)\frac{\partial u}{\partial x} + A_8(t,x)u + f(t,x) \tag{II}$$

where $u(t,x) = col(u_1(t,x),\ u_2(t,x),\dots,u_n(t,x))$ is an unknown function vector, $A_i(t,x)$ is $n\times n$-matrices, $i=\overline{1,8}$, and $f(t,x)$ is an n-function vector continuous in the domain Ω.

In the first subsection, the existence and uniqueness of the solution of the Goursat problem for a system of equations (II)

$$u(t,0) = \psi_1(t), \quad t\epsilon[0,T], \tag{19}$$
$$u(0,x) = \varphi_1(x), \quad x\epsilon[0,\omega], \tag{20}$$
$$\left.\frac{\partial^2 u(t,x)}{\partial x \partial t}\right|_{x=0} = \psi_2(t), \quad t\epsilon[0,T], \tag{21}$$
$$\left.\frac{\partial^3 u(t,x)}{\partial t \partial x^2}\right|_{t=0} = \varphi_2(x), \quad x\epsilon[0,\omega], \tag{22}$$

with conditions (19)-(22) are investigated. In the course of the study of the introduction of new unknown functions, the original problem was reduced to an equivalent Goursat-type problem for a system of second-order hyperbolic equations and integral relations. Taking into account the initially set conditions, the solution of the Goursat problem is determined using the method of sequential approximation. Having put the solution of the Goursat problem into integral relations, the solution of the originally given problem is obtained. Theorems on the convergence of approximate solutions and its uniqueness are proved.

In the second subsection at

$$A_1(t,x) = A_2(t,x) = A_6(t,x) = A_7(t,x) = A_8(t,x) = 0,$$

the existence and uniqueness of the solution of the initial periodic problem for the system of equations (II)

$$u(0,x) = \varphi(x), \quad x \in [0,\omega], \tag{23}$$
$$\left.\frac{\partial u(t,x)}{\partial t}\right|_{t=0} = \left.\frac{\partial u(t,x)}{\partial t}\right|_{t=T}, \quad x \in [0,\omega], \tag{24}$$
$$u(t,0) = \psi_1(t), \quad t \in [0,T], \tag{25}$$
$$\left.\frac{\partial u(t,x)}{\partial x}\right|_{x=0} = \psi_2(t), \quad t \in [0,T], \tag{26}$$

with conditions (23)-(26) is investigated. To this goal, new functions are introduced, and this problem is reduced to an equivalent periodic problem for a system of partial derivatives of second-order integro-differential equations and integral relations. Taking into account the periodicity condition and introducing a functional parameter, the problem under consideration proceeds to a problem consisting of two parts. While one part is a Goursat problem for a system of hyperbolic equations, the other part is a Cauchy problem for differential equations. Taking into account the initial conditions and applying the method of sequential approximation, approximate solutions are obtained by putting two problems in a queue. Theorems on the convergence of approximate solutions of this problem and its uniqueness are proved.

In the third subsection at

$$A_1(t,x) = A_2(t,x) = A_3(t,x) = A_6(t,x) = A_7(t,x) = 0,$$

the existence and uniqueness of the solution of the initial boundary value problem for the system of equations (II)

$$u(0,x) = \varphi_1(t), \qquad x \in [0,\omega], \tag{27}$$

$$\sum_{i=0}^{m} \left\{ P_i(x) \frac{\partial^2 u(t_j,x)}{\partial x^2} + S_i(x) \frac{\partial^2 u(t_j,x)}{\partial t^2} \right\} \Bigg|_{t=t_i} = \varphi_2(x), \quad x \in [0,\omega], \tag{28}$$

$$u(t,0) = \psi_1(t), \;\; t \in [0,T], \tag{29}$$

$$\left. \frac{\partial^2 u(t,x)}{\partial x \partial t} \right|_{x=0} = \psi_2(t), \qquad t \in [0,T], \tag{30}$$

with conditions (27)-(30) is investigated. To do this, the problem under consideration is reduced to an equivalent problem for integro-differential equations of the second order, with the introduction of an unknown function. The introduction of the functional parameter is reduced to two problems, which are a Goursat-type problem for a system of hyperbolic equations and a Cauchy problem for differential equations. Taking into account the initial conditions and using the method of sequential approximation, two problems are solved sequentially and approximate solutions are found. Theorems on the convergence of approximate solutions of this problem and its uniqueness are proved. An example was compiled where it was established that the conditions of the uniqueness theorem of the solution are fulfilled and it is shown that the solution is explicitly.

In the third chapter, a system of fourth-order partial differential equations in the domain $\Omega = [0,T] \times [0,\omega]$ is considered

$$\frac{\partial^4 u}{\partial t^3 \partial x} = A_1(t,x) \frac{\partial^3 u}{\partial t^2 \partial x} + A_2(t,x) \frac{\partial^3 u}{\partial t^3} + A_3(t,x) \frac{\partial^2 u}{\partial t^2} + A_4(t,x) \frac{\partial^2 u}{\partial t \partial x} +$$
$$+ A_5(t,x) \frac{\partial u}{\partial t} + A_6(t,x) \frac{\partial u}{\partial x} + A_7(t,x) u + f(t,x). \tag{III}$$

where $u(t,x) = col(u_1(t,x),\ u_2(t,x), \dots, u_n(t,x))$ is an unknown vector-function; $A_s(t,x)$, $(s = \overline{1,7})$, $n \times n$-matrices and $f(t,x)$ is an n - vector-function continuous in the domain Ω.

In the first subsection, the existence and uniqueness of the solution of the initial boundary value problem for the system of equations (III)

$$\sum_{j=1}^{m}\left\{\sum_{i=1}^{3}\left[P_{i,j}(x)\frac{\partial^i u(t_j,x)}{\partial t^{i-1}\partial x}+S_{i,j}(x)\frac{\partial^i u(t_j,x)}{\partial t^i}\right]+ +L_j(x)u(t,x)\right\}\Big|_{t=t_j}=\varphi_1(x), \quad (31)$$

$$\frac{\partial u(t,x)}{\partial t}\Big|_{t=0}=\varphi_2(x), \qquad u(0,x)=\varphi_3(x), \qquad x\in[0,\omega], \quad (32)$$

$$u(t,0)=\psi(t), \qquad t\in[0,T] \quad (33)$$

with conditions (31)-(33) are investigated. The Goursat problem is equivalent to a system of three integral equations. The convergence of approximate solutions of problems and their uniqueness is established by the method of sequential approximation using the initial given ones. The existence and uniqueness of the solution of the problem under study are formulated and proved in the form of theorems. The Goursat problem is equivalent to a system of three integral equations. The convergence of approximate solutions of problems and their uniqueness is established by the method of sequential approximation using the initial given ones. The existence and uniqueness of the solution of the problem under study are formulated and proved in the form of theorems.

In the second subsection, the existence and uniqueness of the solution of the initial multipoint problem for the system of equations (III)

$$u(0,x)=\varphi_1(x), \;\; x\in[0,\omega], \quad (34)$$

$$\frac{\partial u(t,x)}{\partial t}\Big|_{t=0}=\varphi_2(x), \qquad x\in[0,\omega], \quad (35)$$

$$\frac{\partial^2 u(t,x)}{\partial t^2}\Big|_{t=0}=\varphi_3(t), \qquad x\in[0,\omega], \quad (36)$$

$$\sum_{j=0}^{m}P_j(t)\frac{\partial^2 u(t,x)}{\partial t^2}\Big|_{x=x_j}=\psi(t), \qquad t\in[0,T] \quad (37)$$

with conditions (34)-(37) are studied. To determine its solution, new unknown functions are introduced and the initial given problem is reduced to equivalent multipoint boundary value problems for a system of partial derivatives of second-order differential equations and integral relations. Having made the substitution again, they proceed to the multipoint problem for partial derivatives of first-order differential equations and integral relations. The solution of a multipoint problem for partial derivatives of first-order differential equations is shown, taking into account the initial conditions. A method of sequential approximation of the solution of a multipoint problem for partial derivatives of first-order differential equations, taking into account the initial conditions, is defined. With the help of the solutions found, the solution of the initial given problem is determined through integral relations. Theorems on the convergence of approximate solutions and its uniqueness are proved.

In the third subsection, the existence and uniqueness of the solution of a semi-periodic problem for a system of equations (III)

$$u(0,x) = \varphi_1(x), \;\; x \in [0,\omega], \tag{38}$$

$$\left.\frac{\partial u(t,x)}{\partial t}\right|_{t=0} = \varphi_2(x), \qquad x \in [0,\omega], \tag{39}$$

$$\left.\frac{\partial^2 u(t,x)}{\partial t^2}\right|_{t=0} = \left.\frac{\partial^2 u(t,x)}{\partial t^2}\right|_{t=T}, \qquad x \in [0,\omega], \tag{40}$$

$$u(t,0) = \psi(t), \;\; t \in [0,T] \tag{41}$$

with conditions (38)-(41) are investigated. Unknown functions are introduced into the initial given problem and reduced to an initial boundary value problem for a system of partial derivatives of second-order integro-differential equations and integral relations. Next, a functional parameter is introduced, the last problem is reduced to the Goursat problem and integral equations. The Goursat problem is equivalent to a system of three integral equations. Using the periodic condition and the parametrization method of D. S. Dzhumabaev, it was shown that there are solutions to the Goursat problem for a system of hyperbolic equations and the Cauchy problem for differential equations. Approximate solutions of the Goursat problem and the Cauchy problem were determined using the algorithm for finding solutions. Theorems on the convergence of approximate solutions and its uniqueness are proved.

Thus, the following new scientific results were obtained in the dissertation work: a constructive method was proposed for the study of initial boundary value problems for the three classes of systems of fourth-order partial differential equations considered, on the basis of which the conditions for unique solvability of the problems were established. The ways of finding approximate solutions to the problems under consideration are given and the conditions of convergence of the constructed algorithms are found. The problems under study are reduced to a non-local boundary value problem for systems of second-order hyperbolic equations and functional relations. Using the conditions of unique solvability of boundary value problems for systems of second-order hyperbolic equations, the conditions of unique solvability of the original problem are set. The convergence conditions of the algorithms for finding a solution provide conditions for the unique solvability of the problems considered simultaneously.

The proposed research method and the assigned results can be applied to initial boundary value problems for systems of high-order partial differential equations, boundary value problems for non-classical differential equations. In addition, the established conclusions can be used in the study of various initial-boundary, non-local problems for classes (I), (II), (III) fourth-order partial differential equations.