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UNPREDICTABLE SOLUTIONS OF DIFFERENTIAL EQUATIONS AND HYBRID SYSTEMS WITH APPLICATIONS FOR NEURAL NETWORKS

ABSTRACT
of the thesis for the degree of Doctor of Philosophy (PhD) in the specialty 6D060100 – Mathematics

Structure and scope of the thesis. The thesis consists of an introduction, 3 chapters (the first chapter includes 3 sections and 4 subsections, the third chapter consists of 3 sections), a conclusion, list of references.

The number of illustrations, tables and references. In the thesis used 19 illustrations, 2 tables and 146 references.

Keywords. Oscillations, unpredictable function, discontinuous unpredictable function, impulsive system, hybrid system, *B*-topology, generalized piecewise constant argument, Hopfield-type neural networks, asymptotic stability.

The actuality of the thesis is due to the numerous applications of differential equations and hybrid systems in solving problems of natural science and the extensive use of neural networks in the modern world. The thesis is based on the concept of unpredictable functions, which were introduced by M. Akhmet and M.O. Fen [1-3]. In the thesis, the existence of continuous and discontinuous unpredictable solutions of impulsive systems and differential equations with generalized piecewise constant argument are proved, and the results are applied to Hopfield-type neural networks. The main results of the thesis have been published in international peer reviewed journals, which confirms the actuality of the research.

The theoretical backgrounds of the present research lie entirely in the theory of dynamical systems founded by H. Poincaré[4] and G. Birkhoff[5]. H. Poincaré determined that underneath of chaotic dynamics is the Poisson stable motion. Poisson stable motions as well as periodic, quasi-periodic and almost periodic ones are crucial for the theory of differential equations.

Recurrent motions and Poisson stable points are at the center of the qualitative theory of motions of dynamical systems. Poisson stable points were considered by A. Poincaré as the main element in describing the complexity of celestial dynamics. More than a century later, improving the Poisson stable point, M. Akhmet and M.O. Fen introduced the concept of an unpredictable point. They have developed the recurrence in functional spaces to its ultimate form, when beside the Poisson stable functions are assigned the *separation* property.

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1 Akhmet M., Fen M.O. Unpredictable points and chaos // Communications in Nonlinear Science and Numerical Simulation. – 2016. – Vol. 40. – P. 1-5.

2 Akhmet M., Fen M.O. Poincare chaos and unpredictable functions // Communications in Nonlinear Science and Numerical Simulation. – 2017. – Vol. 48. – P. 85-94.

3 Akhmet M., Fen M.O. Non-autonomous equations with unpredictable solutions // Communications in Nonlinear Sciences and Numerical Simulation. – 2018. – Vol. 59.– P. 657-670.

4 Poincaré H. Les methodes nouvelles de la mecanique celeste, - Paris: Gauthier-Villars, 1892,1899. - Vol. 1-3.

5 Birkhoff, G.D. Dynamical Systems. – USA: American Mathematical Society, 1927. – 305 p.

An unpredictable trajectory is necessarily positively Poisson stable, and one of its distinctive features is the emergence of chaos in the corresponding quasi-minimal set. The chaos based on the presence of an unpredictable trajectory is called *Poincaré chaos*. That is, the proof of unpredictability simultaneously verifies the Poincaré chaos of the Bebutov dynamics in the functional space with the topology of convergence on compact sets of the real axis. This opens new prospects for control and synchronization of chaos in differential equations.

Unpredictable points and unpredictable functions are becoming increasingly applicable in the study of chaos theory. For instance, some topological properties of Poincaré chaos were considered by A. Miller[6], and R. Thakur and R. Das[7,8]. Moreover, they are used not only the definition of the unpredictable point and Poincaré chaos, but also a technique based on Poisson and separation sequences. Recently it is proved that unpredictable motions also present in random processes.

Hybrid systems are a concept in dynamical systems theory and have important applications. A model is called *hybrid* if it combines continuous and discrete dynamics. Impulsive systems and differential equations with generalized piecewise constant argument are essentially hybrid equations, that is, a combination of continuous and discrete equations.

Impulsive systems describe dynamics of real-world phenomena in which abrupt interruptions of continuous processes are present, and they play a crucial role in various fields such as mechanics, medicine, neural networks, communication systems. Existence of periodic and almost periodic solutions of linear nonhomogeneous systems perturb by Dirac's delta functions was investigated by A. Halanay and D. Wexler[9]. But in case of nonlinear systems, this method of research leads to significant difficulties. One approach to solving this problem was to use a *jump operator* in the system. This approach is used in the research.

The most developed theory of differential equations with generalized piecewise constant functions as arguments, have been introduced by M. Akhmet[10-12]. The proposals became most general not only in modeling, but also very powerful in methodological sense, since the equivalent integral equations have been suggested to open the research gate for methods of operators' theory and functional analysis.

The novelty of our results has to be considered with respect to oscillations and modeling for neural networks. Neural networks have special significance in adaptive

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6 Miller A. Unpredictable points and stronger versions of Ruelle–Takens and Auslander–Yorke chaos // Topology and its Applications. – 2019. - № 253. – P. 7–16.

7 Thakur R., Das R. Strongly Ruelle–Takens, strongly Auslander–Yorke and Poincaré chaos on semiflows // Communications in Nonlinear Science and Numerical Simulation. – 2020. - № 81:105018.

8 Thakur R., Das R. Sensitivity and chaos on product and on hyperspatial semiflows // Journal of Difference Equations and Applications. – 2021. – Vol. 27(1). – P. 1-15.

9 Halanay A., Wexler D. Teoria calitativà a sistemelor cu impulsuri. – Bucharest: Editura Academiei, 1968. –312 p.

10 Akhmet M.U. On the integral manifolds of the differential equations with piecewise constant argument of generalized type // Proceedings of the Conference on Differential and Difference Equations at the Florida Institute of Technology. – 2005. – P. 11-20.

11Akhmet M.U. On the reduction principle for differential equations with piecewise constant argument of generalized type // Journal of Mathematical Analysis and Applications. – 2007. – Vol.336, №1. – P. 646-663.

12 Akhmet M.U. Stability of differential equations with piecewise constant arguments of generalized type // Nonlinear Analysis: Theory, Methods & Applications. – 2008. – Vol.68, № 4. – P. 794-803.

pattern recognition, vision, image processing, associative memory, enhancement of X-Ray and computed tomography images. The models considered in the thesis are based on unpredictable perturbations, which allow studying the chaotic signals in neural networks.

The aim of the thesis research is to investigate linear, quasilinear impulsive systems and quasilinear differential equations with generalized piecewise constant argument that have unpredictable perturbations, using the method and theoretical basis laid down in the works M. Akhmet and M.O. Fen. Furthermore, the study of the presence of unpredictable oscillations in Hopfield-type neural networks.

The research problems:

a) obtain sufficient conditions for the existence and uniqueness of an asymptotically stable discontinuous unpredictable solution of linear and quasilinear impulsive systems;

b) prove the existence and uniqueness of an exponentially stable unpredictable solution of quasilinear differential equations with a generalized piecewise constant argument;

c) establishment sufficient conditions for the existence of asymptotically stable strongly unpredictable continuous oscillations in in Hopfield-type neural networks;

d) investigate the asymptotic stability of discontinuous unpredictable oscillations in an impulse neural network with a Hopfield structure;

e) determine sufficient conditions for the existence and uniqueness of exponentially stable unpredictable oscillations in Hopfield neural networks with a generalized piecewise constant argument

f) presentation of numerical examples, simulations and block diagrams confirming the validity of the theoretical results.

The research methods. In the thesis, methods and results of the theory of functional analysis, algebra and differential equations are widely used.

The objects of research are the continuous and discontinuous unpredictable oscillations of impulsive systems, differential equations with generalized piecewise constant argument and Hopfield-type neural networks.

The scientific novelties. This is a first time impulsive systems and differential equations with generalized piecewise constant argument that have unpredictable perturbations have been introduced and investigated. As well as the models Hopfield-type neural networks described by the differential equations and hybrid systems have been researched.

The novelties of the thesis are as follows:

a) sufficient conditions for the existence and uniqueness of asymptotically stable discontinuous unpredictable solutions of linear and quasilinear impulsive differential equations are obtained;

b) the existence and uniqueness of exponentially stable unpredictable solutions of quasilinear differential equations with generalized piecewise constant argument are proved;

c) sufficient conditions for the existence and uniqueness of asymptotically stable strongly unpredictable oscillations in the Hopfield type model of the neural networks are established;

d) the existence of asymptotically stable discontinuous unpredictable oscillations of the impulsive model of neural networks with the Hopfield structure are investigated;

e) sufficient conditions for the existence and uniqueness of exponentially stable unpredictable oscillations of Hopfield-type neural networks with generalized piecewise constant argument are determined;

f) the numerical examples, simulations and block diagrams confirming the feasibility of the theoretical results are provided.

The results of the thesis which are taken out on defense:

- the concept of discontinuous unpredictable function, as well as the ways to construction discontinuous unpredictable functions;
- sufficient conditions for existence of unique asymptotically stable discontinuous unpredictable solutions of linear impulsive systems;
- conditions for existence of unique asymptotically stable discontinuous unpredictable solutions of quasilinear impulsive systems;
- conditions for existence of unique of exponentially stable unpredictable solutions of quasilinear differential equations with generalized piecewise constant argument;
- sufficient conditions for the existence of asymptotically stable strongly unpredictable oscillations of Hopfield-type neural networks;
- sufficient conditions for the existence of asymptotically stable discontinuous unpredictable oscillations of impulsive neural networks with the Hopfield structure;
- sufficient conditions for the existence and uniqueness of exponentially stable unpredictable solutions of Hopfield-type neural networks with generalized piecewise constant argument.

The personal contribution of the author. All the results of the thesis are obtained by the author. The participation of co-authors and scientific consultants consists in setting goals and discussing the results.

Approbation of the received results. The main results of the thesis were reported and discussed at the following events:

- V International Scientific and Practical Conference IICT MES RK "Informatics and Applied Mathematics" (Almaty, Kazakhstan, September 29-October 1, 2020);
- International Conference on Artificial Intelligence and Applied Mathematics in Engineering (ICAIAME 2020) (Antalya, Turkey, October 9-11, 2020);
-Traditional International April Scientific Conference, in honor of the Day of Science Workers of the Republic of Kazakhstan (Almaty, Kazakhstan, April 5-8, 2021);
- Scientific seminar "The Problems of Applied Mathematics and Computer Science", Department of Mathematics, K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan (seminar leader Doctor of Physical and Mathematical Sciences, professor Zh. Sartabanov).

Publications. On the topic of the dissertation, 7 articles were published, including 3 publications in a ranking scientific journal indexed in the Scopus database, 1 publication in scientific journal included in the list recommended by the Committee for Control in the Sphere of Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan for publication of the main scientific results of scientific activities, 3 publications in the materials of the international conferences, including 1 publication in the materials of a foreign conference indexed in the database Scopus.

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