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STUDY THE EXISTENCE OF SOLUTIONS OF INHOMOGENEOUS SYSTEMS SUCH AS CLAUSEN

ABSTRACT

of the thesis for the degree of Doctor of Philosophy (PhD) in the specialty 6D060100 – Mathematics

Relevance of the topic. The study is dedicated to new effective methods for the existence of solutions to inhomogeneous generalized hypergeometric equations of high order and systems of combined equations. Methods studying the existence of solutions to higher-order hypergeometric equations and inhomogeneous systems of nonlinear differential equations of the second and third order have not been considered in earlier works. Hypergeometric functions obtained using equations of this kind and their systems are widely used in electrodynamics, nuclear and mathematical physics, and radio electronics. From such important usage, in general, arises the necessity for comprehensive investigation of the problems related to the existence of solutions to inhomogeneous generalized hypergeometric equations and systems of equations.

Special functions encountered in applications of mathematical physics are autonomous cases of the hypergeometric Gauss function $F(\alpha,\beta,\gamma;x)$. If the right side of the inhomogeneous Bessel equation is given in the form of power or linear functions, then the solutions of the inhomogeneous equation are special functions of Lommel, Struve, Anger, Weber. Inhomogeneous equations of hypergeometric origin, which are solutions of these and other well-known special functions have been studied in A.V.Babistr's monograph. The monograph examines inhomogeneous ordinary differential equations of the second order, but there is no methodology for their complex study.

Later generalized hypergeometric equations and systems of the third and higher order, associated with the study of multidimensional derivative equations, have been more frequently used in various problems. Although these works focus on the search for solutions of homogeneous equations, nonhomogeneous cases are not considered.

The first example of a generalized hypergeometric function is the Clausen function (1828), which depends on five parameters. However, the study of the properties of the Clausen function, which is a solution of a simple third-order differential equation, has not reached its full extent. The search for solutions of nonhomogeneous Clausen differential equations, used in particular in problems of science and engineering, has not been fully investigated. The peculiarities of constructing solutions of derivative equations obtained by transforming the Clausen

equation have also not been considered. The same situation can be stated for generalized hypergeometric systems of two partial derivative differential equations.

Objective of the work. To study the problem of existence of solutions of the Clausen-type systems, to find efficient methods for constructing nonhomogeneous equations and solutions of the Clausen-type systems, and to extend these methods to systems of equations whose solutions are generalized hypergeometric functions with multiple variables. To develop a theory that utilizes efficient methods for constructing normal and normal-regulator solutions in the domain of singular curves.

Research objectives:

a) to demonstrate the features of constructing solutions to systems of nonhomogeneous generalized differential equations and autonomous differential equations consisting of two equations, using undetermined coefficients and improved Frobenius-Latyshew methods, to classify distinctive points and curves;

b) to establish effective methods for constructing solutions to the nonhomogeneous Claussen equation and neighborhoods of singular points $x=0$ and $x=\infty$;

c) to establish an effective method for constructing solutions to the derived nonhomogeneous Claussen equation;

d)to show the features of constructing solutions to systems consisting of two partial differential equations of third order near regular and irregular curves;

e) to demonstrate the possibilities of using generalized hypergeometric functions in constructing solutions to the system of equations of generalized hypergeometric origin in the form of a multidimensional orthogonal polynomial;

f) to construct the general solution of the nonhomogeneous simple Claussen system and investigate the properties of the solution;

g) to study the peculiarities of constructing solutions to nonhomogeneous fundamental and derived Claussen systems;

h) to study the existence of a normal-regulatory solution of the derivative system obtained by transitioning from the Lauricella system to a limit, and their properties;

t) to demonstrate the peculiarities of the relationship between the Khudozhnikov function and the normal-regulatory solutions.

Research Object. Generalized hypergeometric equations of non-homogeneous higher order and systems of second and third order differential equations with autonomous derivatives, Clausen's equation and Clausen's systems, which are their special cases.

Research Methods. The main method used to construct efficient algorithms for normal, normal-regulator, and finite solutions of systems of differential equations with autonomous derivatives consisting of two or three equations is the improved Frobenius-Latysheva method. Improved and modified methods by P. Appell, V. Vilchinsky, Ch. Hermite, E. AINS, A. Erdéy, K. Latysheva for cases of systems of partial derivatives of differential equations used in the analytical theory of ordinary differential equations were also applied.

When studying special functions with multiple variables and systems with orthogonal polynomial solutions, the works of E. Kummer, J. Camp de Ferry, P. Appell, Ch. Hermite, M. Lorichella, V.I. Khudozhnikova, Clausen were used.

Research Subject. The creation of an effective method for finding solutions of non-homogeneous generalized hypergeometric type equations of higher order and systems of autonomous derivative third-order differential equations in the form of generalized hypergeometric functions, justification of conditions for the existence of solutions.

Scientific research news:

a) The Frobenius-Latysheva method and undetermined coefficients were used in the dissertation to study previously unexplored systems of non-homogeneous partial differential equations of the third order.

b) Necessary conditions for the existence of normal and normal-regular solutions of the system of non-homogeneous partial differential equations of the third order have been developed, as well as efficient methods for their construction.

c) Features of constructing solutions of the non-homogeneous Clausen simple system near singular curves have been identified, and properties of the corresponding solutions have been determined.

d) Conditions for the existence of solutions of the non-homogeneous main and derived Clausen systems have been determined, and efficient methods for finding solutions have been shown.

e) The theorems regarding the existence of normal-regular solutions of the Khudozhnikov system, derived from the transition from the Lauricella system to the limit case, have been proven for the cases of a general variable n.

f) Connections between the Khudozhnikov function and normal-regular solutions created within the framework of the research have been identified.

Results presented for defense:

- Construction of solutions of non-homogeneous generalized differential equations using the method of undetermined coefficients and the right-hand sides of equations in a system of autonomous differential equations depending on the peculiarities of their transmission.
- Application of the Frobenius-Latysheva method to the study of systems of partial differential equations of the third order.
- Construction of solutions of the generalized system of third-order equations using the Frobenius-Latysheva method in the vicinity of various unique curves.
- Determination of specific types of equations and systems of equations whose solutions are generalized hypergeometric functions, and construction of solutions using the Camp de Ferrier method.
- Necessary conditions for the existence of a system of homogeneous self-contained differential equations of the third order and normal $(0,0)$ and (∞,∞) normal-regular solutions near a singular curve.
- Construction of solutions of non-homogeneous simple, main, and derived

Clausen systems in the form of generalized hypergeometric functions.

- Establishment of connections between normal-regular solutions of systems consisting of two, three, and n equations, derived using the Frobenius-Latysheva method, and the Khudozhnikov function.

Reliability and validity of the obtained results. The dissertation extensively utilizes methods and results from the theory of differential equations and systems of autonomous differential equations, the solutions of which are special functions with one or several variables. The scientific results are formulated in the form of lemmas and theorems. The significance, reliability of the research results, and the obtained results are summarized in publications in highly demanded journals.

Theoretical and practical significance of the research. The theoretical assessment of the results obtained in the dissertation is distinguished by the development of the analytical theory of systems of autonomous third-order differential equations and finding solutions to the discussed problems in the form of generalized hypergeometric functions. Therefore, the practical significance of the dissertation lies in its important place in the theory of multivariate special functions and its application in research on various problems in mathematical physics, electrodynamics, multidimensional equation theory, radio electronics, and antenna theory.

Author's personal contribution. All the results presented in the dissertation were obtained personally by the author or with their direct participation. Scientific advisors and co-authors contributed only to the formulation of the problem and the discussion of the obtained results.

The obtained results were presented and discussed at the following seminars and conferences:

- Traditional International April Scientific Conference (Institute of Mathematics and Mathematical Modeling, Ministry of Education and Science of the Republic of Kazakhstan, Almaty, Kazakhstan, April 3-5, 2019).
- International Scientific Conference "Theoretical and Applied Problems of Mathematics, Mechanics, and Informatics" (E.A. Buketov Karaganda State University, Karaganda, Kazakhstan, June 12-14, 2019).
- "Current Problems of Analysis, Differential Equations, and Algebra" (EMJ-2019): an international conference dedicated to the 10th anniversary of the journal "Eurasian Mathematical Journal" (L.N. Gumilyov Eurasian National University, Nur-Sultan, Kazakhstan, October 16-19, 2019).
- Scientific seminar "Complex Analysis and Its Applications", Tajik National University, Dushanbe, Tajikistan, November 2020 (seminar led by Dr.N. Radzhabov).
- International Scientific and Practical Conference "Problems of Fundamental and Applied Mathematics at the Present Stage" (Moscow State University Branch in Kazakhstan, Nur-Sultan, Kazakhstan, June 4, 2021).
- International Scientific Conference "Nonlocal Boundary Problems and Related Problems in Mathematical Biology, Informatics, and Physics" (Nalchik,

Russia, December 5-9, 2021).

- IX International Scientific Conference "Problems of Differential Equations, Analysis, and Algebra" (Aktobe, Kazakhstan, May 24-28, 2022).

- Scientific seminar "Problems of Applied Mathematics and Informatics", Department of Mathematics, K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan (seminar led by Dr. Z. Sartabanov).

Publications. The main scientific conclusions of the dissertation topic have been published in 23 scientific works. Among them, 2 articles have been published in reputable scientific journals indexed in the Scopus database, 3 articles in publications recommended by the Committee for Control in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan, 1 article in a scientific journal of the Republic of Kazakhstan, 1 article in the journal based on the scientific citation of the Russian index, and 16 articles in the proceedings of international scientific conferences in Kazakhstan and neighboring countries.

Structure and Volume of the Dissertation. The dissertation consists of an introduction, three chapters, a conclusion, and a list of references. The conclusions and formulas are numbered with two indexes. The first index represents the section number, and the second index represents the formulation number of the formulas in that section. The total volume of the dissertation is 127 pages. The list of references consists of 117 titles.

The first section of the work examines the possibilities of constructing solutions in the form of generalized hypergeometric functions

$$x^{B+1}\left(\mu_{B+1}-\lambda_{B+1}x^k\right)\frac{d^{B+1}y}{dx^{B+1}}+...+x\left(\mu_1-\lambda_1 x^k\right)\frac{dy}{dx}+\left(\mu_0-\lambda_0 x^k\right)y=f(x)$$

for various independent cases of a generalized differential equation of a nonhomogeneous hypergeometric type

$$_AF_B\left(a_1,a_2,...,a_A;b_1,b_2,...,b_B;x\right)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n...(a_A)_n}{(b_1)_n(b_2)_n...(b_B)_n n!}x^n$$

the distinguishing points of these cases are regular or irregular, where $\mu_i,\lambda_i(i=0,1,...,B+1)$ – represents unknown constants, k – represents an integer, $f(x)$ – represents a generalized power series, $a_k, k=\overline{1,A}$ and $b_j, j=\overline{1,B}$ – represents constant parameters. The first example of such hypergeometric functions can be called the Clausen-type function

$$_3F_2\left(a_1,a_2,a_3;b_1,b_2;x\right)=F\begin{pmatrix}a_1, & a_2, & a_3\\ b_1, & b_2 & \end{pmatrix}x=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n n!}x^n$$

which depends on five parameters.

In section 1.1, the main definitions and concepts used in the research work are given.

Next, data on how to construct generalized differential equations are presented

$$x^3\left(\mu_3-\lambda_3x^k\right)\frac{d^3y}{dx^3}+x^2\left(\mu_2-\lambda_2x^k\right)\frac{d^2y}{dx^2}+x\left(\mu_1-\lambda_1x^k\right)\frac{dy}{dx}+\left(\mu_0-\lambda_0x^k\right)y=0$$

a third-order differential equation is introduced, where, by determining the unknown coefficients using the Kampe de Feriet method, the Clausen equation is obtained in the form

$$x^2(1-x)\frac{d^3y}{dx^3}+[1+b_1+b_2-(3+a_1+a_2+a_3)x]x\frac{d^2y}{dx^2}+$$
$$+[b_1b_2-(1+a_1+a_2+a_3+a_1a_2+a_2a_3+a_1a_3)x]\frac{dy}{dx}-a_1a_2a_3y=0$$

the solution of which is the Clausen function.

In subsection 1.2, the features of constructing solutions to nonhomogeneous generalized differential equations with singular points of the regulator using the method of undetermined coefficients are considered. The main properties of the Clausen function are formulated in the internal subsection. The theorems about the existence of both independent and general solutions, necessary conditions for the existence of a solution, as well as the existence of a set of regular solutions in the vicinity of singular points $x=0$ and $x=\infty$, are proven for the Clausen equation.

In subsection 1.3, the possibilities of studying the solutions of the nonhomogeneous Clausen equation depending on the peculiarities of transferring the function $f(x)$ on the right side of the equation are considered. In the case when the right hand side of the nonhomogeneous equation is represented in the form $f(x)=x^\rho$, the specificity of using the Frobenius-Latysheva method to find its general solution is shown.

The second part of the dissertation is devoted to the study of the nonhomogeneous system and the homogeneous system that is obtained when the right side of this nonhomogeneous system is $f_i(x,y)=0,\ (i=1,2)$.

Several theorems about the existence of solutions to the nonhomogeneous system are proven in subsection 2.1. The peculiarities of finding solutions to the nonhomogeneous hypergeometric second-order system of origin are considered in the subsection. Nonhomogeneous systems whose solutions are orthogonal polynomials are poorly studied. Therefore, as an example, a theorem is formulated and proven for the nonhomogeneous state of the Hermitian system.

In subsection 2.2, the issues of constructing solutions to this system of homogeneous equations are considered in the presence of $\omega = 2$. In this case, it is necessary to determine the existence of solutions to the third-order system. It is important to consider the possibilities of creating solutions to the system that can be obtained with different values of $h = 0, h = 1$ and $h \geq 2$.

The autonomous case $h = 1$ was previously considered, and systems of the Kampé de Fériet kind were obtained. The main concepts of the Frobenius-Latysheva method, which is the main method of conducting research, are included in the internal subsection. These include the concept of the characteristic Frobenius function, the concepts of the system of determining equations determined by the signs $(0,0)$ and (∞, ∞).

In subsection 2.3, the general properties of the nonhomogeneous third-order system obtained in the value of $\omega = 2$ are proven.

In subsection 2.3.1, the theorems are presented for the case of $h = 1$. In the general case $k \geq 2$, the proofs are also carried out in a similar manner. In this case, the independent solutions are expressed in the form of $Z = Z(x^k, y^k), (k \geq 2)$.

Subsection 2.3.2 discusses the peculiarities of applying the method of undetermined coefficients when the right side of the nonhomogeneous system, obtained in value $\omega = 2$, is represented in the form of a generalized series with two variables. The independent solution is sought in the form of a generalized series near the singular curve $(0,0)$.

In subsection 2.3.3, the peculiarities of constructing solutions to a nonhomogeneous autonomous third-order derivative equation obtained by adding two homogeneous, nonhomogeneous, and homogeneous equations using the autonomous solution method by Camp de Férrieres are shown. The Frobenius-Latysheva and undetermined coefficients methods are applied to prove the given theorems.

Subsection 2.4 considers the simple Claussin system and its solution properties. I. Gorn has defined that the products of two hypergeometric functions, each of which is a hypergeometric function with only one variable, also belong to hypergeometric functions of the second order. They will be solutions of a system consisting of two Claussin equations. Such systems are called simply Claussin-type systems. Several theorems about the properties of the homogeneous and nonhomogeneous simple Claussin system are proven.

The third part of the dissertation is entirely dedicated to studying the existence of normally regular solutions of a system of differentially generated equations with a self-derivative of the second order. Such normally regular solutions are solutions of a homogeneously generated system obtained by transitioning from the Lavricheva system to a limit. Solutions of this system in the form of new functions were defined by V.I. Khudozhnikov. The dissertation work shows that besides the solutions of the investigated system in the form of Khudozhnikov's function, there also exist normally regular solutions. As a visual example, a system of equations consisting of two and three equations is proposed, with relationships established between the functions. The obtained results are generalized for systems consisting of n equations.

We are mainly interested in functions, caused by several variables related to the function

$$F_B\left(\begin{matrix}(\alpha_n), & (\beta_n) \\ \gamma & \end{matrix}\middle|(z_n)\right)=\sum_{m_1,\dots m_n}\frac{(\alpha_1)_{m_1}\dots(\alpha_n)_{m_n}(\beta_1)_{m_1}\dots(\beta_n)_{m_n}}{(\gamma)_{m_1+\dots+m_n}}\frac{z_1^{m_1}}{m_1!}\dots\frac{z_n^{m_n}}{m_n!}$$

Lauricella, where $|z_k|<1, k=\overline{1,n}$. The function F_B is an autonomous solution of the Lauricella type (F_B) system .

$$z_i(1-z_i)\frac{\partial^2 w}{\partial z_i^2}+\sum_{j=1,j\neq i}^{n} z_j\frac{\partial^2 w}{\partial z_j\partial z_i}+[\gamma-(\alpha_i+\beta_i+1)z_i]\frac{\partial w}{\partial z_i}-\alpha_i\beta_i w=0,\quad i=\overline{1,n}$$

By multiple transition from the function Lauricella F_B to the limit V.I. Khudozhnikov introduced a new function of the type

$$\Phi_{B,n}^{k,l}\left(\begin{matrix}(\alpha_k), & (\alpha_l^{'}), & (\beta_k) \\ & \gamma & \end{matrix}\middle|(z_n)\right)=\sum_{i_1,\dots i_n}\frac{\Pi(\alpha_k)_{i_k}\cdot(\beta_k)_{i_k}}{(\gamma)_{\sum i_n}}\cdot\Pi(\alpha_1^{'})_{i_{l+k}}\cdot\Pi\frac{(z_n)^{i_n}}{i_n!}$$

where the following short notations were used

$$(a_n)=(a_1,\dots,a_n),(z_n)=(z_1,\dots,z_n),\ \Pi(\alpha_n)_{i_n}=\prod_{k=1}^{n}(a_k)_{i_k},\ \sum i_n=\sum_{k=1}^{n}i^k,\ \sum i_1,\dots,i_n=\sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\dots\sum_{i_n=0}^{\infty}(\dots)$$

Along with the newly emerged function introduced by V.I. Khudozhnikov, a normal-regulator solution of type

$$W(z_1,z_{2,\dots,}z_n)=\exp(\alpha_{1,0,\dots,0}z_1+\alpha_{0,1,\dots,0}z_2+\dots+\alpha_{0},_{\dots,1}z_n)\times$$
$$\times z_1^{\rho_1}z_2^{\rho_2}\dots z_n^{\rho_n}\sum_{m_1,\dots,m_n=0}^{\infty}A_{m_1,m_{2,\dots},\dots,m_n}z_1^{m_1}z_2^{m_2}\dots z_n^{m_n},\quad A_{0,\dots,0}\neq 0$$

has been introduced, obtained through a limit transition, where $\rho_t (t=\overline{1,n})$, $A_{m_1,m_2,...m_n} (m_1,...,m_n = 0,1,2,...)$, $\alpha_{1,0,...,0}, \alpha_{0,1,0,...,0}, ..., \alpha_{0,...,0,1}$ –unknown constants.

In subsection 3.1.1, the resulting hypergeometric system of type

$$z_i(1-z_i)\frac{\partial^2 w}{\partial z_i^2} + \sum_{j=1, j\neq i}^{n} z_j \frac{\partial^2 w}{\partial z_j \partial z_i} + [\gamma - (\alpha_i + \beta_i + 1)z_i]\frac{\partial w}{\partial z_i} - \alpha_i \beta_i w = 0, \quad i=\overline{1,k}$$

$$\sum_{j=1}^{n} z_j \frac{\partial^2 w}{\partial z_j \partial z_i} + (\gamma - z_i)\frac{\partial w}{\partial z_i} - \alpha'_{i-k} w = 0, \quad i=\overline{k+1,k+l}$$

$$\sum_{j=1}^{n} z_j \frac{\partial^2 w}{\partial z_j \partial z_i} + \gamma \frac{\partial w}{\partial z_i} - w = 0, \quad i=\overline{k+l+1,n}$$

is presented, obtained through multiple transitions from the Lauricella system (F_B) to a limit, where the notations $(a_n)=(a_1,a_2,...,a_n),(z_n)=(z_1,z_2,...,z_n)$ were used.

The existence of normal regular solutions of autonomous conditions of the given combined system and their connection with Khudzhitov's functions have been studied for the case $n=2$, consisting of two equations. The main theorem here is related to the consideration of the Gorne system (Φ_2).

In subsection 3.2, the possibilities of constructing normal-regular solutions of the inhomogeneous hypergeometric system are examined and several theorems are presented.

The theorems discussed in subsection 3.3 are generalized to the case of n variables.