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Unpredictable oscillations of differential equations and neural networks

Annotation of the thesis for the degree of Doctor of Philosophy (PhD) in the specialty 6D060100 – Mathematics

Structure and scope of the thesis. The thesis consists of an introduction, 2 chapters (the first chapter includes 4 sections, the second chapter consists of 3 sections), a conclusion and list of references.

Number of illustrations, tables, and references used. In the thesis used 112 references and 15 illustrations.

Keywords. Oscillations, unpredictable function, strongly unpredictable function, linear differential equations, quasilinear differential equations, neural networks, shunting inhibitory cellular neural networks, inertial neural networks, asymptotical stability.

The actuality of the topic is due to the numerous applications of differential equations in solving problems of natural science and the widespread use of neural networks in the modern world. The thesis is based on the concepts of unpredictable functions, which were introduced by M. Akhmet and M.O. Fen[1−4]. We proved the existence of unpredictable solutions of differential equations, and considered the problem of unpredictable oscillations in neural networks. The main results of the thesis have been published in international peer reviewed journals, which confirms the actuality of the research.

The thesis researches. The thesis is devoted to the study of unpredictable

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1 Akhmet M., Fen M.O. Unpredictable points and chaos // Communications in Nonlinear Science and Numerical Simulation. – 2016. - № 40. – P. 1-5.
2 Akhmet M., Fen M.O. Existence of unpredictable solutions and chaos // Turkish Journal of Mathematics. – 2017. -vol. 41. - №. 2.- P. 254–256.
3 Akhmet M., Fen M.O. Poincare chaos and unpredictable functions // Communications in Nonlinear Science and Numerical Simulation. - 2017. – № 48. - P. 85-94.
4 Akhmet M., Fen M. Non-autonomous equations with unpredictable solutions // Communications in Nonlinear Sciences and Numerical Simulation. – 2018. – №59. – P. 657-670.

solutions of differential equations and unpredictable oscillations of neural networks.

Preliminaries. Oscillations are necessary attribute of various processes occurring in nature. Of exceptional theoretical and practical importance are the oscillatory motions described by differential equations. In the literature a large number of results have been obtained for periodic, quasiperiodic, and almost periodic solutions of differential equations due to the established mathematical methods and important applications. On the other hand, recurrent and Poisson stable solutions are also crucial for the theory of differential equations.

The founders of the theory of non-linear oscillations are H. Poincare[5–6] and A.M. Lyapunov[7], who created a mathematical apparatus suitable for the study of nonlinear systems. The theory of non-linear dynamics focuses mainly on periodic motions. The first functions which can be considered still as "periodic" and sufficiently determined for strict mathematical analysis were quasiperiodic functions introduced and investigated by P. Bohl and E. Esclangon independently. The fundamental papers of H. Bohr provided the theory of almost periodic functions, which we call as H. Bohr almost periodic functions nowadays. Then different approaches to almost periodicity were found by N.N. Bogolyubov, A.S. Besicovitch, S. Bochner, V.V. Stepanov, and others. The almost periodic functions are of great importance for development of harmonic analysis on groups, Fourier series and integrals on groups. In the paper, published by S. Bochner provided extension of the theory of almost periodic functions with values in a Banach space. The first paper on the existence of almost periodic solutions was written by H. Bohr and O. Neugebauer, and nowadays the theory of almost periodic equations has been developed in connection with problems of differential equations, stability theory, and dynamical systems. The list of the applications of the theory has been essentially extended and includes not only ordinary differential equations and classical dynamical systems, but also, wide classes of partial differential equations and equations in Banach spaces. The concepts of recurrent

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5 Poincaré H. Les methodes nouvelles de la mecanique celeste, - Paris: Gauthier-Villars, 1892. - Vol. 1,2.
6 Poincaré H. Les methodes nouvelles de la mecanique celeste. - Paris, 1899. - vol. III; reprint (Dover, New York, 1957).
7 Lyapunov A.M. The General Problem of the Stability of Motion (In Russian). - Doctoral dissertation, University Kharkov, 1892, English translations: Stability of Motion. - New-York & London: Academic Press, 1966.

motions and Poisson stable points are classical notions central to the qualitative theory of motions for dynamical systems. Poisson stable points was considered by H. Poincare as the main element in the description of complexity in celestial dynamics.

The foundation of the research of non-linear oscillations in Kazakhstan were laid by V.H. Kharasakhal and O.A. Zhautykov. D.U. Umbetzhanov and his colleagues, intensively investigated almost periodic and multi-periodic solutions for differential and evolution systems. Nowadays, Kazakhstan mathematicians continue to make significant contribution to the analysis of oscillations.

In recent decades, researchers have focused on studying oscillations in neural networks. Neural networks were created to study brain activity. There are many models of neural networks, which mathematically described by recurrent and differential equations. For example, Hopfield type neural networks, shunting inhibitory cellular neural networks, Cohen-Grossberg neural networks, inertial neural networks are investigated.

Oscillatory neural networks are effective for image recognition, and in activating network states associated with memory recall. It is natural that neural oscillations became the core of interdisciplinary research that unites neuroscience, psychophysics, biophysics, cognitive psychology, and computational modeling.

This is why, many researchers are studying periodic, almost periodic, and exponential stability for neural networks considering input/output mechanisms.

Recently, the study of chaotic oscillations in neural networks starts to be of significant interest. Solutions of the chaotic systems are irregular, and this is reflected by data related to experiments and observations.

A few years ago, M. Akhmet and M.O. Fen introduced the concept of unpredictable point and unpredictable function and thereby significantly expanded the boundaries of the classical theory of dynamical systems, founded by H. Poincare and G. Birkhoff. An unpredictable point is the modernization of the Poisson stable point. They proved that the quasi-minimal set is chaotic set, if the Poisson stable point also admit the unpredictability property. Thus, the presence of chaos in a dynamic system is determined by the presence of only one point - unpredictable. The unpredictable points are used by A. Miller[8], R. Thakur and R. Das[9] in topological spaces where they considered Poincare chaos, strongly Ruelle-Takens chaos, and strongly Auslander-York chaos. Unpredictable functions were defined as unpredictable points in the Bebutov dynamical system with the only difference that the topology of convergence on compact sets of the real axis is used instead of the metric space. The use of such convergence makes it possible to significantly simplify the problem of proving the existence of unpredictable solutions for differential equations. And one can completely remain in the field of the theory of differential equations without mentioning the original or related results in the theory of dynamical systems or chaos.

The goal of this study. The goal of the thesis is to use the method and theoretical basis laid down in the articles by M. Akhmet and M.O. Fen, to investigate linear, quasilinear differential equations with unpredictable perturbations. And also use them for study unpredictable oscillations of SICNNs and INNs.

The scientific novelties. The novelties of the thesis are as follows:

a) the existence and uniqueness of asymptotically stable unpredictable and strongly unpredictable solutions of differential equations are shown;

b) the existence of asymptotically stable unpredictable and strongly unpredictable oscillations of the neural networks are proved;

c) the examples and numerical simulations confirming the feasibility of the theoretical results are given.

The results of the thesis which are taken out on defense:

- the theorem on the existence and uniqueness of uniformly asymptotically stable unpredictable solutions of linear differential equations;
- the theorem on the existence and uniqueness of uniformly exponentially stable unpredictable solutions of quasilinear differential equations;
- the theorem on the existence of uniformly exponentially stable unpredictable solutions of SICNNs;

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8 Miller A. Unpredictable points and stronger versions of Ruelle–Takens and Auslander–Yorke chaos // Topology and its Applications. – 2019. - № 253. – P. 7–16.

9 Thakur R., Das R. Strongly Ruelle–Takens, strongly Auslander–Yorke and Poincaré chaos on semiflows // Communications in Nonlinear Science and Numerical Simulation. – 2020. - № 81:105018.

- the theorem on the existence and uniqueness of asymptotically stable strongly unpredictable unpredictable solutions of SICNNs;
- the theorem on the existence and uniqueness of asymptotically stable unpredictable solutions of INNs;
- the ways to construction unpredictable functions.

Research methods. In the thesis, methods and results of the theory of functional analysis, algebra and differential equations are widely used.

The theoretical and practical significance of the results. The scientific significance of the study lies in the fact that the results obtained will become the basis for the study of unpredictable oscillations of various differential equations, including impulse differential equations, partial differential equations. The control of unpredictable oscillations will allow them to be used in medicine, biology, cryptography and many other fields.

Publications. 11 publications have been published on the topic of the dissertation, of which 5 articles in rating scientific journals indexed in the Scopus database, 2 articles in publications recommended by the CCSES MES RK, the rest in the materials of international conferences, including 1 publication in the materials of foreign conferences (Scopus).

Summary of work.

The chapter 1 devoted to unpredictable solutions of differential equations.

In the first section the basic definitions of unpredictable functions are presented.

Definition 1. *A uniformly continuous and bounded function* $\vartheta: \mathbb{R} \to \mathbb{R}^p$ *is unpredictable if there exist positive numbers* ε_0, δ *and sequences* $\{t_n\}, \{u_n\}$ *both of which diverge to infinity such that* $\|\vartheta(t+t_n) - \vartheta(t)\| \to 0$ *as* $n \to \infty$ *uniformly on compact subsets of* $\mathbb{R}$ *and* $\|\vartheta(t+t_n) - \vartheta(t)\| \geq \varepsilon_0$ *for each* $t \in [u_n - \delta, u_n + \delta]$ *and* $n \in \mathbb{N}$.

The convergence of the sequence $\vartheta(t+t_n)$ is said to be *Poisson stability of the unpredictable function* or simply *Poisson stability* as well as existence of the numbers

ε_0, δ and the sequence u_n is allow *the unpredictable property of the unpredictable function*.

The Definition 1 implies that some coordinates may not be unpredictable scalar valued functions. But it is very important for applications to consider movements that are unpredictable in all dimensions, that is, *strongly unpredictable functions*. Therefore, we introduced the following definition:

Definition 2. *A uniformly continuous and bounded function* $\vartheta: \mathbb{R} \to \mathbb{R}^p$, $\vartheta = (\vartheta_1, \vartheta_2, \ldots, \vartheta_p)$, *is strongly unpredictable if there exist positive numbers* ε_0, δ *and sequences* $\{t_n\}, \{u_n\}$, *both of which diverge to infinity such that* $\vartheta(t + t_n) \to \vartheta(t)$ *as* $n \to \infty$ *uniformly on compact sets of* $\mathbb{R}$ *and* $|\vartheta_i(t + t_n) - \vartheta_i(t)| \geq \varepsilon_0$ *for all* $i = 1,2\ldots, p$, $t \in [u_n - \delta, u_n + \delta]$, *and* $n \in \mathbb{N}$.

The properties of unpredictable functions are given. The example of the unpredictable function is constructed.

In the second section proved existence of uniformly asymptotically stable unpredictable solution of linear differential equations:

$$x'(t) = Ax(t) + g(t), \tag{1}$$

where $x \in \mathbb{R}^p$ and the function $g: \mathbb{R} \to \mathbb{R}^p$ is unpredictable. And all eigenvalues of the constant matrix $A \in \mathbb{R}^{p \times p}$ have nonzero real parts.

The following theorem is true:

Theorem. *System (1) possesses a unique unpredictable solution. Additionally, if all eigenvalues of the matrix A have negative real parts, then the unpredictable solution is uniformly asymptotically stable.*

The main object *of the third section* is the system of quasilinear differential equations:

$$x'(t) = Ax(t) + f(x(t)) + g(t), \tag{2}$$

where $x(t) \in \mathbb{R}^p$, p is a fixed natural number, the function $f: \mathbb{R}^p \to \mathbb{R}^p$ is continuous in all of its arguments, and all eigenvalues of the constant matrix $A \in \mathbb{R}^{p \times p}$ have nonzero real parts. Moreover, the function $g: \mathbb{R} \to \mathbb{R}^p$ is unpredictable.

It is known that one can find a regular $p \times p$ matrix B such that the transformation $x = By$ reduces the system (2) to the system

$$y'(t) = Cy(t) + F(y) + G(t), \quad (3)$$

where $C = B^{-1}AB, F(y) = B^{-1}f(By)$, and $G(t) = B^{-1}g(t)$. In system (3), the matrix C is of the form $\mathrm{diag}(C_-, C_+)$, where the eigenvalues of the $q \times q$ matrix C_- and $(p-q) \times (p-q)$ matrix C_+ respectively have negative and positive real parts.

One can confirm that there exist numbers $K \geq 1$ and $\alpha > 0$ such that $||e^{C_- t}|| \leq Ke^{-\alpha t}$ for all $t \geq 0$ and $||e^{C_+ t}|| \leq Ke^{\alpha t}$ for all $t \leq 0$.

The following conditions are required.

(C1) There exists a positive number L_f such that $||f(x_1) - f(x_2)|| \leq L_f ||x_1 - x_2||$ for all $x_1, x_2 \in \mathbb{R}^p$.

(C2) $\frac{2}{\alpha} K(||B|| ||B^{-1}|| L_f + 1) < 1$;

The next theorem concerning the unpredictable solution of the system (2) is proved.

Theorem. *Suppose that conditions $(C1)-(C2)$ are valid then system (2) possesses a unique unpredictable solution. Moreover, the unpredictable solution is uniformly exponentially stable if all eigenvalues of the matrix A have negative real parts.*

In the fourth section we extend Definitions 1 and 2 to the class of functions with several independent variables. The following new definitions have been introduced.

Definition 3. *A continuous and bounded function* $f(t, x): \mathbb{R} \times G \to \mathbb{R}^p$, $f =$

$= (f_1, f_2, \ldots, f_p)$, $G \subset \mathbb{R}^p$ *is a bounded domain, is unpredictable in* t *if it is uniformly continuous in* t *and there exist positive numbers* ε_0, δ *and sequences* $\{t_n\}, \{u_n\}$, *both of which diverge to infinity such that* $\sup_G ||f(t + t_n, x) - f(t, x)|| \to 0$ *as* $n \to \infty$ *uniformly on compact sets in* $\mathbb{R}$ *and* $||f(t + t_n, x) - f(t, x)|| \geq \varepsilon_0$ *for* $t \in [u_n - \delta, u_n + \delta]$, $x \in G$ *and* $n \in \mathbb{N}$.

Definition 4. *A continuous and bounded function* $f(t, x)\colon \mathbb{R} \times G \to \mathbb{R}^p$, $f = (f_1, f_2, \ldots, f_p)$, $G \subset \mathbb{R}^p$ *is a bounded domain, is strongly unpredictable in* t *if it is uniformly continuous in* t *and there exist positive numbers* ε_0, δ *and sequences* $\{t_n\}$, $\{u_n\}$ *both of which diverge to infinity such that* $\sup_G \left|\left|f(t + t_n, x) - f(t, x)\right|\right| \to 0$ *as* $n \to \infty$ *uniformly on compact sets in* $\mathbb{R}$ *and* $|f_i(t + t_n, x) - f_i(t, x)| \geq \varepsilon_0$ *for all* $i = 1,2\ldots, p$, $(t, x) \in [u_n - \delta, u_n + \delta] \times G$, *and* $n \in \mathbb{N}$.

Considered the following system of differential equations

$$x'(t) = Ax(t) + f(t, x), \tag{4}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^p$, p is a fixed natural number, all eigenvalues of the constant matrix $A \in \mathbb{R}^{p \times p}$ have negative real parts, $f\colon \mathbb{R} \times G \to \mathbb{R}^p, f = (f_1, f_2, \ldots, f_p)$, $G = \{x \in \mathbb{R}^p, ||x|| < H\}$, where H is a positive number. It is true that there exist two real numbers $K \geq 1$ and $\gamma < 0$ such that $||e^{At}|| \leq Ke^{\gamma t}$ for all $t \geq 0$.

The following conditions will be needed:

$(C1)$ the function $f(t, x)$ is strongly unpredictable in the sense of Definition 4. The Definition 4 implies that there exists a positive number M such that $\sup_{\mathbb{R} \times G} ||f(t, x)|| = M < \infty$;

$(C2)$ there exists a positive constant L such that the function $f(t, x)$ satisfies the inequality $||f(t, x_1) - f(t, x_2)|| \leq L||x_1 - x_2||$ for all $t \in \mathbb{R}, x_1, x_2 \in G$;

$(C3)$ $\gamma < -\dfrac{KM}{H}$;

$(C4)\ \gamma < -KL.$

The next theorem is proved.

Theorem. *If conditions* $(C1) - (C4)$ *are fulfilled, then the system (4) admits a unique uniformly exponentially stable strongly unpredictable solution.*

Moreover, it is proved that if the function $f(t, x)$ is unpredictable in the sense of Definition 3, then the system (4) admits a unique uniformly exponentially stable unpredictable solution.

The theoretical results confirmed by examples and graphical illustrations.

In the chapter 2 we discuss about unpredictable oscillations in neural networks.

In the first section considered SICNNs, which have been introduced by A. Bouzerdoum and R. Pinter[10]. In its original formulation, the SICNNs model is a two-dimensional grid of processing cells. Let C_{ij} denote the cell at the (i, j) position of the lattice. Denote by $N_r(ij)$ the r – neighborhood C_{ij}, such that

$$N_r(ij) = \{C_{kp} \colon \max(|k-i|, |p-j|) \le r, 1 \le k \le m, 1 \le p \le n\},$$

where m and n are fixed natural numbers. In SICNNs, neighboring cells exert mutual inhibitory interactions of the shunting type. The dynamics of the cell C_{ij} is described by the following nonlinear ordinary differential equation,

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} f(x_{kp}(t))x_{ij}(t) + v_{ij}(t), \qquad (0.5)$$

where x_{ij} is activity of the cell C_{ij}, the constant a_{ij} represents the passive decay rate

10 Bouzerdoum A., Pinter R. Shunting inhibitory cellular neural networks: derivation and stability analysis // IEEE Transactions on Circuits and Systems I Fundamental Theory and Applications. – 1993. - № 40. – P. 215–21.

of the cell activity, $C_{ij}^{kp} \geq 0$ is the connection or coupling strength of postsynaptic activity of the cell C_{kp} transmitted to the cell C_{ij} and the activation $f(x_{kp})$ is a positive continuous function representing the output or firing rate of the cell C_{kp}, $v_{ij}(t)$ is the external input to the cell C_{ij}.

Let us denote by $\mathcal{A}$ the set of functions $u(t) = (u_{11}, \dots, u_{1n}, \dots, u_{m1} \dots, u_{mn})$, $t, u_{ij} \in \mathbb{R}, i = 1,2,\dots,m, j = 1,2,\dots,n$, where $m, n \in \mathbb{N}$, such that:

($\mathcal{A}1$) functions $u(t)$ are uniformly continuous and there exists a positive number H such that $||u||_1 < H$ for all $u(t) \in \mathcal{A}$;

($\mathcal{A}2$) there exists a sequence $t_p, t_p \to \infty$ as $p \to \infty$ such that for each $u(t) \in \mathcal{A}$ the sequence $u(t + t_p)$ uniformly converges to $u(t)$ on each closed and bounded interval of the real axis.

The following assumptions will be needed

$(C1)$ the function $v(t) = (v_{11}, \dots, v_{1n}, \dots, v_{m1} \dots, v_{mn}), t, v_{ij} \in \mathbb{R}, i = 1,2, \dots, m$, $j = 1,2,\dots,n$, in system (5) belongs to $\mathcal{A}$ and is unpredictable such that there exist positive numbers $\delta, \varepsilon_0 > 0$ and a sequence $t_p \to \infty$ as $p \to \infty$, which satisfy $||v(t + t_p) - v(t)|| \geq \varepsilon_0$ for all $t \in [s_p - \delta, s_p + \delta]$, and $p \in \mathbb{N}$.

$(C2)$ for the rates we assume that $\gamma = \min_{(i,j)} a_{ij} > 0$ and $\bar{\gamma} = \max_{(i,j)} a_{ij}$;

$(C3)$ there exist positive numbers m_{ij} and m_f such that $\sup_{t \in \mathbb{R}} |v_{ij}| \leq m_{ij}$ for all $i = 1, \dots, m, j = 1, \dots, n$, and $\sup_{|s|<H} |f(s)| \leq m_f$;

$(C4)$ there exists Lipschitz constant L such that $|f(s_1) - f(s_2)| \leq L|s_1 - s_2|$ for all $s_1, s_2, |s_1| < H, |s_2| < H$;

$(C5)\left(LH + m_f\right) \max_{(i,j)} \sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} < \gamma$ for all $i = 1, \dots, m, j = 1, \dots, n$.

The main result of the present section is mentioned in the next theorem.

Theorem. *Suppose that conditions* $(C1) - (C5)$ *are valid, then the system (5) possesses a unique asymptotically stable unpredictable solution.*

In the second section we consider following SICNNs

$$\frac{dx_{ij}}{dt} = -b_{ij}x_{ij} - \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp} f(x_{kp}(t))x_{ij}(t) + g_{ij}(t), \tag{6}$$

with strongly unpredictable perturbations.

We denote by $\mathcal{B}$ the set of functions $u(t) = (u_{11}, \dots, u_{1n}, \dots, u_{m1} \dots, u_{mn})$, $t, u_{ij} \in \mathbb{R}, i = 1,2,\dots,m, j = 1,2,\dots,n,$ where $m, n \in \mathbb{N}$, such that:

($\mathcal{B}$1) functions $u(t)$ are uniformly continuous;

($\mathcal{B}$2) there exists a positive number H such that $||u||_1 < H$ for all $u(t) \in \mathcal{B}$;

($\mathcal{B}$3) there exists a sequence $t_p, t_p \to \infty$ as $p \to \infty$ such that for each $u(t) \in \mathcal{B}$ the sequence $u(t + t_p)$ uniformly converges to $u(t)$ on each closed and bounded interval of the real.

The following conditions are needed

$(D1)$ the function $g(t) = (g_{11}, \dots, g_{1n}, \dots, g_{m1} \dots, g_{mn}), t, g_{ij} \in \mathbb{R}, i = 1,2, \dots, m,$ $j = 1,2,\dots,n,$ in system (0.6) belongs to $\mathcal{B}$ and is strongly unpredictable such that there exist positive numbers $\delta, \varepsilon_0 > 0$ and a sequence $t_p \to \infty$ as $p \to \infty$, which satisfy $|g_{ij}(t + t_p) - g_{ij}(t)| \geq \varepsilon_0$ for all $t \in [s_p - \delta, s_p + \delta], i = 1,\dots,m, j = 1,\dots,n,$ and $p \in \mathbb{N}$.

$(D2)$ $\gamma \leq b_{ij} \leq \bar{\gamma}$, where $\gamma, \bar{\gamma}$ are positive numbers;

$(D3)$ $|g_{ij}(t)| \leq m_{ij}$, where m_{ij} are positive numbers, for all $i = 1, \dots, m, j = 1, \dots, n,$ and $t \in \mathbb{R}$;

$(D4)$ $|f(s)| \leq m_f$, for $|s| < H$ and some constant $m_f > 0$;

$(D5)$ there exists Lipschitz constant L such that $|f(s_1) - f(s_2)| \leq L|s_1 - s_2|$ for all $s_1, s_2, |s_1| < H, |s_2| < H$;

$(D6)$ $m_f \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp} < b_{ij}$, for each $i = 1, \dots, m, j = 1, \dots, n$;

$(D7)$ $\dfrac{m_{ij}}{b_{ij} - m_f \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}} < H$, for all $i = 1, \dots, m, j = 1, \dots, n;$

$(D8)$ $(LH + m_f) \max_{(i,j)} \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp} < \gamma.$

The following theorem is proved.

Theorem. *Suppose that conditions $(D1)$ -$(D8)$ are valid, then the system (6) possesses a unique asymptotically stable strongly unpredictable solution.*

In the third section the following INN is considered:

$$\frac{d^2x_i(t)}{dt^2} = -a_i \frac{dx_i(t)}{dt} - b_i x_i(t) + \sum_{j=1}^{p} c_{ij} f_j(x_j(t)) + v_i(t), \tag{7}$$

where $t, x_i \in \mathbb{R}, i = 1,2,\dots,p, p$ denotes the number of neurons in the network; $x_i(t)$ with $i = 1,2,\dots,p,$ corresponds to the state of the unit i at time t; the second derivative is called an inertial term; $b_i > 0, a_i > 0$ are constants; f_i with $i = 1,2,\dots,p,$ denote the measures of activation to its incoming potentials of ith neuron; c_{ij} for all $i,j = 1,2,\dots,p,$ are constants, which denote the synaptic connection weight of the unit j on the unit i; $v_i(t)$ are external inputs on the ith neuron at time t.

We assume that the coefficients c_{ij} are real, the activation functions $f_i: \mathbb{R} \to \mathbb{R}$ are continuous functions satisfy the following condition:

(I1) $|f_i(x_1) - f_i(x_2)| \le L_i|x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}$, where $L_i > 0$ are Lipschitz constant, for all $i = 1,2,\dots,p,$ and $\max_{1\le i\le p} L_i = L.$

By introducing the following variable transformation

$$y_i(t) = \xi_i \frac{dx_i(t)}{dt} + \zeta_i x_i(t), i = 1, \dots, p, \tag{8}$$

the neural network (7) can be written as

$$\frac{dx_i(t)}{dt} = -\frac{\zeta_i}{\xi_i} x_i(t) + \frac{1}{\xi_i} y_i(t), \tag{9}$$

$$\frac{dy_i(t)}{dt} = -\left(a_i - \frac{\zeta_i}{\xi_i}\right) y_i(t) - \left(\xi_i b_i - \zeta_i \left(a_i - \frac{\zeta_i}{\xi_i}\right)\right) x_i(t) + $$
$$+\xi_i \sum_{j=1}^{p} c_{ij} f_j(x_j(t)) + \xi_i v_i(t), \tag{10}$$

Denote by Σ the set of vector-functions, $\varphi(t) = (\varphi_1, \varphi_2, \ldots, \varphi_{2p})$, such that:

(K1**)** functions $\varphi(t)$ are uniformly continuous;

(K2) there exists a positive number H such that $||\varphi||_1 < H$ for all $\varphi(t) \in \Sigma$;

(K3) there exists a sequence, $t_n \to \infty$ as $n \to \infty$ such that $\varphi(t + t_n)$ uniformly converges to $\varphi(t)$ on each bounded interval of the real line.

The following conditions will be needed:

(I2) the functions $v_i(t), i = 1, \ldots, p,$ in system (7) are unpredictable;

(I3) there exists a positive number H and M_f such that $|f_i(s)| \le M_f, i = 1, \ldots, p,$
$|s| < H.$

Moreover, we assume that for positive real numbers ζ_i and ξ_i, $i = 1, \ldots, p$ the following inequalities are valid:

(I4) $a_i > \frac{\zeta_i}{\xi_i} + \xi_i, \zeta_i > \xi_i > 1;$

(I5) $(a_i - \frac{\zeta_i}{\xi_i}) - (|\zeta_i \left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + \xi_i) > 0;$

(I6) $\dfrac{\xi_i M_f \sum_{j=1}^{p} c_{ij}}{(a_i - \frac{\zeta_i}{\xi_i}) - (|\zeta_i \left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + \xi_i)} < H;$

(I7) $\frac{1}{(a_i - \frac{\zeta_i}{\xi_i})}(|\zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + L\xi_i \sum_{j=1}^{p} c_{ij}) < 1;$

(I8) $\max_i(\frac{1}{\xi_i}, |\zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + L\xi_i \sum_{j=1}^{p} c_{ij}) < \min_i\left(\frac{\zeta_i}{\xi_i}, a_i - \frac{\zeta_i}{\xi_i}\right).$

The following theorem is proved.

Theorem. *Assume that conditions* $(I1) - (I8)$ *are fulfilled. Then the system (7) admits a unique asymptotically stable unpredictable solution.*

Illustrative examples with graphs confirming the obtained theoretical results for unpredictable oscillations of neural networks are presented.

Thus, in the dissertation, in the investigation of unpredictable oscillations of differential equations and neural networks, the following results were obtained:

- some properties of unpredictable functions are shown and an example of unpredictable is constructed;
- the existence and uniqueness of uniformly asymptotically stable unpredictable solution of linear differential equations are proved;
- the conditions for the existence and uniqueness of uniformly exponentially stable unpredictable solution of quasilinear differential equations are obtained;
- proved the existence and uniqueness of an asymptotically stable unpredictable and strongly unpredictable solution of SICNNs;
- sufficient conditions for the existence and uniqueness of an asymptotically stable unpredictable solution to the INN are investigated;
- created ways to construction unpredictable functions.

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