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Qualitative properties and numerical solution of a nonlinear boundary value problem for the Fredholm integro-differential equation

Annotation of the thesis for the degree of Philosiphy Doctor (PhD) in the specialty 6D060100 — Mathematics

Structure and scope of the dissertation. The dissertation work consists of an introduction, three sections (the first section includes 3 subsections, the second and third sections include four subsections each), a conclusion, a list of references and an appendix.

Number of illustrations, tables, and references used. The list of used references consists of 116 titles.

Keywords. Nonlinear Fredholm integro-differential equation, Volterra integro-differential equation, nonlinear boundary value problem for integro-differential equation, parameterization method, special Cauchy problem, regular partition, iterative process, isolated solution, algorithm, numerical solution.

The relevance of the topic is due, on the one hand, to the numerous applications of integro-differential equations in solving problems of natural sciences and, on the other hand, to the necessity of development of new constructive methods that allow effectively determine the solvability of nonlinear problems for integro-differential equations and find their solutions.

Fredholm integro-differential equations have several of features that should be taken into account at statement of problems for these equations and development of methods for solving them.

In particular, a linear nonhomogeneous Fredholm integro-differential equation might be unsolvable without additional requirements imposed on the solution[1],[2]. Note that the criteria for the solvability and unique solvability of linear boundary value problems for the Fredholm integro-differential equations were obtained quite recently[3]. Main existing methods for studying the boundary value problems for the Fredholm integro-differential equations, such as the A.I. Nekrasov's method and the Green's function method, are applicable under unique solvability of some intermediate problems. These methods establish various sufficient conditions for the existence of solution, but do not allow us to obtain criteria for the solvability of

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[1] Boichuk A.A., Samoilenko A.M. Generalized inverse operators and Fredholm boundary value problems. – Berlin, De Gruyter, 2016.

[2] Dzhumabaev D.S. On one approach to solve the linear boundary value problems for Fredholm integro-differential equations // J. Comput. Appl. Math. – 2016. – Vol. 294.

[3] Dzhumabaev D.S. A Method for Solving the Linear Boundary Value Problem for an Integro Differential Equation // Comput. Math. Math. Phys. – 2010 – Vol. 50, № 7.

boundary value problems for these equations. Therefore, based on the parameterization method[4] D.S. Dzhumabaev was proposed a new approach to the study and solving the boundary value problems for the Fredholm integro-differential equations. Partition of the interval, where the Fredholm integro-differential equation is considered, puts a special Cauchy problem in accordance with this equation. If the latter problem is uniquely solvable, then its solution can be represented via the parameters introduced and the known values of the integro-differential equation. Substitution of these expressions into the boundary condition and solution continuity conditions at the interior points of the partition constructs a system of linear algebraic equations in the parameters introduced. It is proved that the solvability of the boundary value problem is equivalent to the solvability of this system. As can be seen, this approach also requires the unique solvability of the intermediate problem, the special Cauchy problem. However, in contrast to the above mentioned methods, for a linear Fredholm integro-differential equation with continuous matrices of the differential part and integral term, there exists an interval partition, in which the special Cauchy problem is uniquely solvable. This property of the intermediate problem of the parameterization method allows us to obtain the criteria for the solvability and unique solvability of linear boundary value problems for the Fredholm integro-differential equations. The partition Δ_N of the interval $[0, T]$, in which the corresponding special Cauchy problem is uniquely solvable, is said to be regular for the Fredholm integro-differential equation under consideration. The Fredholm integro-differential equation may not have a solution. Due to the existence of unsolvable Fredholm integro-differential equations, the classical general solution does not exist for all Fredholm integro-differential equations. Therefore, Dzhumabaev was introduced a new general solution[5] of the linear Fredholm integro-differential equation. A new general solution exists for any linear Fredholm integro-differential equation. The application of this solution allows us to establish the solvability criteria for a linear inhomogeneous integro-differential Fredholm equation and boundary value problems for this equation.

Mainly the nonlinear problems are solved by iterative methods. Many efficient iterative methods, such as Newton's method, require choosing a "good" initial guess. In[6], iterative processes were constructed for nonlinear equations with unbounded operators and conditions for their convergence were set. The results were applied to

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[4] Dzhumabaev D.S. Conditions for Unique Solvability of a Linear Boundary Value Problem for an Ordinary Differential Equation // Zh. Vychisl. Mat. Mat. Fiz. – 1989. – № 29.

[5] Dzhumabaev D.S. New general solutions to linear Fredholm integro-differential equations and their applications on solving the boundary value problems // J. Comput. Appl. Math. – 2018. – Vol. 327.

[6] Dzhumabaev D.S. Convergence of iterative methods for unbounded operator equations // Mat. Zametki. – 1987. – Vol. 41, № 5.

the nonlinear boundary value problems for the ordinary differential equations and partial differential equations[7] .

At applying the parameterization method to a boundary value problem for a nonlinear Fredholm integro-differential equation, the special Cauchy problem for a system of nonlinear integro-differential equations with parameters is the intermediate problem. In this case, iterative methods are used both for solving the special Cauchy problem and for solving the systems of nonlinear algebraic equations in parameters. An approach to finding an initial guess for the solutions to these problems is offered.

One of the most effective methods for solving the problems for integro-differential equations with a small numerical parameter is the averaging method[8], which allows reducing the solvability of a boundary value problem for integro-differential equations to the solvability of a similar problem for a differential averaged system.

The purpose of the research is to develop constructive methods for the investigating and solving initial and boundary value problems for the Volterra and Fredholm integro-differential equations.

The object of research are the initial and boundary value problems for nonlinear integro-differential equations.

Research methods. The methods and results of the theory of differential, integro-differential and operator equations are used in the dissertation work. The main method of research and solving the problems considered in the present dissertation is the parameterization method.

Scientific novelty and practical value of the work. In the dissertation work,

- The special Cauchy problem for a system of nonlinear integro-differential equations with parameters is solved;
- The new general solution to the Fredholm integro-differential equation with a nonlinear differential part is constructed;
- The parameterization method is extended to the nonlinear boundary value problems for the Fredholm integro-differential equation;
- An algorithm for finding a solution to a nonlinear boundary value problem for an integro-differential equation is developed and numerically implemented;
- The conditions for the existence of a solution to a boundary value problem for an integro-differential equation under condition of solvability of the averaged boundary value problem for a system of differential equations are established.

The results of the dissertation are mainly theoretical. The scientific significance of the work is to create a constructive method for the studying and solving the

[7] Джумабаев Д.С. Скорость сходимости итерационных процессов для неограниченных операторных уравнений // Известия академия наук Каз ССР. – 1988. – № 5.

[8] Митропольский Ю.А., Байнов Д.Д., Милушева С.Д. Применение метода усреднения для решения краевых задач для обыкновенных дифференциальных уравнений и интегро-дифференциальных уравнений // Мат.физика. – 1979. – Вып.25. – С. 3-22.

problems for nonlinear integro-differential equations. The results obtained in this work can be used for solving the boundary value problems for the Fredholm integro-differential equations, as well as for reading elective courses at the mathematical faculties of universities.

Provisions for the protection. The following are taken out for the defense:

- sufficient conditions for the existence of solutions to the special Cauchy problem for systems of nonlinear integro-differential equations with parameters;
- iterative methods for solving the special Cauchy problem for systems of nonlinear integro-differential equations with parameters and their numerical implementations;
- Δ_N general solution to the Fredholm integro-differential equation with nonlinear differential part, its properties;
- parameterization method for solving a nonlinear boundary value problem for the Fredholm integro-differential equation;
- algorithms for solving the nonlinear boundary value problems for the Fredholm integro-differential equations, their numerical implementations;
- sufficient conditions for the existence of an isolated solution to a nonlinear boundary value problem for the Fredholm integro-differential equation;
- construction of a system of nonlinear algebraic equations in parameters for a boundary value problem for the Fredholm integro-differential equation with a nonlinear differential part and an algorithm for finding its solution;
- algorithms for finding the initial guesses of solutions to the nonlinear special Cauchy problem and to the constructed system of nonlinear algebraic equations;
- justification of the averaging method to investigating the existence of solutions to the initial and boundary value problems for the nonlinear Volterra integro-differential equation.

Publications. The main results of the dissertation work were published in 14 papers, 2 articles in the journals recommended by CCSES MES RK, 1 article in a journal from the Scopus list, 1 article in a journal from the Web of Science list, and 1 article in a journal from the ZbMath list, the rest were published in the materials of international scientific conferences.

Summary of work

In the first Section, consider the Fredholm integro-differential equation with nonlinear differential part

$$\frac{dx}{dt} = f(t,x) + \sum_{k=1}^{m} \varphi_k(t) \int_0^T \psi_k(\tau)x(\tau)d\tau, \quad t \in [0,T], \quad x \in \mathbb{R}^n. \quad (1)$$

By the Dzhumabaev parameterization method, equation (1) is reduced to the system of nonlinear integro-differential equations with parameters

$$\frac{du_r}{dt} = f(t, u_r + \lambda_r) +$$

$$\sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)\left[u_j(\tau) + \lambda_j\right] d\tau, \quad t \in [t_{r-1}, t_r), \tag{2}$$

subject to the initial conditions

$$u_r(t_{r-1}) = 0, \quad r = \overline{1, N}. \tag{0.3}$$

Problem (2), (3) is said to be the special Cauchy problem for a system of nonlinear integro-differential equations with parameters.

Given a vector $\lambda^{(0)} = \left(\lambda_1^{(0)}, \lambda_2^{(0)}, \dots, \lambda_N^{(0)}\right) \in \mathbb{R}^{nN}$ and a positive number $\rho > 0$, we compose the set $G^0(\rho) = \{(t, x): t \in [0, T], \|x - x_0(t)\| < \rho\}$, where $x_0(t) = \lambda_r^{(0)}$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$, and $x_0(T) = \lambda_N^{(0)}$.

Condition 1. *Let the following inequalities be fulfilled:*

(1) $\|f(t, x)\| \le M_0$, $(t, x) \in G^0(\rho)$, M_0 *is constant;*

(2) $M_1 \bar{h} = \left[M_0 + K_0\left(\rho + \left\|\lambda^{(0)}\right\|\right)\right]\bar{h} < \rho$,

where

$$K_0 = \sum_{k=1}^{m} \max_{t \in [0, T]} \|\varphi_k(t)\| \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \|\psi_k(\tau)\| d\tau.$$

We introduce the following sets:

$$G_p^0(\rho) = \left\{(t, x): t \in \left[t_{p-1}, t_p\right), \|x - x_0(t)\| < \rho - M_1\left(t_p - t\right)\right\}, \quad p = \overline{1, N-1},$$

$$G_N^0(\rho) = \{(t, x): t \in [t_{N-1}, t_N], \|x - x_0(t)\| < \rho - M_1(t_N - t)\}, \text{ и}$$

$$G^0(\Delta_N, \rho) = \bigcup_{r=1}^{N} G_r^0(\rho).$$

While solving the boundary value problem for equation (1), we use the limit values of solutions to problem (2), (3), $\lim_{t \to t_r - 0} u_r(t, \lambda)$, $r = \overline{1, N}$. In this connection, we consider the following special Cauchy problem on closed subintervals:

$$\frac{dv_r}{dt} = f(t, v_r + \lambda_r) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)\left[v_j(\tau) + \lambda_j\right] d\tau, \quad t \in [t_{r-1}, t_r], \tag{4}$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1, N}. \tag{5}$$

The special Cauchy problem as the Cauchy problem for the Fredholm integro-differential equations is not always solvable. Therefore, we investigate the solvability of the special Cauchy problem (4), (5).

We choose the numbers $\rho_\lambda = \rho - M_1\bar{h}.$, $\rho_v = M_1\bar{h}$, and construct the sets

$S(\lambda^{(0)}, \rho_\lambda) = \left\{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N) \in \mathbb{R}^{nN}: \left\|\lambda_r - \lambda_r^{(0)}\right\| < \rho_\lambda, \quad r = \overline{1,N}\right\}.$

$S(0, \rho_v) = \left\{v[t] \in \tilde{\mathbb{C}}([0,T], \Delta_N, \mathbb{R}^{nN}): \|v[\cdot]\|_3 < \rho_v\right\}.$

Let us introduce the following notation:

$G(\Delta_N) = \left(G_{p,k}(\Delta_N)\right)$ is the $nm \times nm$ matrix consisting of the $n \times n$ matrices

$$G_{p,k}(\Delta_N) = \sum_{r=1}^{N} \int_{t_{r-1}}^{t_r} \psi_p(\tau) \int_{t_{r-1}}^{\tau} \varphi_k(s) ds d\tau, \quad p,k = \overline{1,m}.$$

$[I - G(\Delta_N)]^{-1} = \left(R_{k,p}(\Delta_N)\right)$, $k,p = \overline{1,m}$, where I is the identity matrix of dimention nm, $R_{k,p}(\Delta_N)$ are the square matrices of dimention n.

Next statement establishes sufficient conditions for the existence of a unique solution to the special Cauchy problem for the system of nonlinear integro-differential Fredholm equations with parameters (4), (5).

Theorem 1. *Let condition 1 be fulfilled, the matrix $I - G(\Delta_N)$ be invertable and the following inequalites be valid:*

(i) $\|f(t,x') - f(t,x'')\| \le L_0\|x' - x''\|$, L_0 *is constant*, $(t,x'), (t,x'') \in G^0(\rho)$;

(ii) $(L_0 + K_0)\bar{h} < 1$;

(iii) $\chi\left(M_0 + K_0\left(\rho_\lambda + \|\lambda^{(0)}\|\right)\right)\bar{h} < \rho_v$, *where*

$$\chi = 1 + \bar{h}\sum_{k=1}^{m} \max_{t\in[0,T]}\|\varphi_k(t)\| \sum_{p=1}^{m} \left\|R_{k,p}(\Delta_N)\right\| \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \left\|\psi_p(s)\right\| ds.$$

Then, for any $\hat{\lambda} \in S(\lambda^{(0)}, \rho_\lambda)$, there exists a unique function system $[t, \hat{\lambda}] = \left(v_1(t,\hat{\lambda}), v_2(t,\hat{\lambda}), \dots, v_N(t,\hat{\lambda})\right)$, the solution to the special Cauchy problem (4), (5) in $S(0, \rho_v)$.

While solving the special Cauchy problem (4), (5) for fixed values of the parameters $\lambda \in \mathbb{R}^{nN}$, we use iterative processes with damping factors.

Given a vector $\lambda^{(0)} = \left(\lambda_1^{(0)}, \lambda_2^{(0)}, \dots, \lambda_N^{(0)}\right) \in \mathbb{R}^{nN}$, a function system $u^{(0)}[t] = \left(u_1^{(0)}(t), u_2^{(0)}(t), \dots, u_N^{(0)}(t)\right) \in \mathbb{C}([0,T], \Delta_N, \mathbb{R}^{nN})$, and some positive numbers ρ, ρ_λ, ρ_u, we compose the folowing sets:

$$S(\lambda^{(0)}, \rho_\lambda) = \left\{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N) \in \mathbb{R}^{nN} \colon \max_{r=\overline{1,N}} \left\|\lambda_r - \lambda_r^{(0)}\right\| < \rho_\lambda\right\},$$

$$S\left(u^{(0)}[t], \rho_u\right) = \left\{u[t] \in \mathbb{C}([0,T], \Delta_N, \mathbb{R}^{nN}) \colon \left\|u[\cdot] - u^{(0)}[\cdot]\right\|_2 < \rho_u\right\},$$

$$S(x_0(t), \rho) = \{x(t) \in \mathbb{PC}([0,T], \Delta_N, \mathbb{R}^n) \colon \|x - x_0\|_5 < \rho\},$$

$$G^0(\rho) = \{(t,x) \colon t \in [0,T], \|x - x_0(t)\| < \rho\},$$

$$G_r^0(\rho) = \{(t,x) \colon t \in [t_{r-1}, t_r), ||x - x_0(t)|| < \rho\}, \quad r = \overline{1,N}\},$$

where the function $x_0(t)$ is defined by the equalities

$$x_0(t) = \lambda_r^{(0)} + u_r^{(0)}(t), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1,N},$$

$$x_0(T) = \lambda_N^{(0)} + \lim_{t \to T-0} u_N^{(0)}(t),$$

and belongs to $\mathbb{PC}([0,T], \Delta_N, \mathbb{R}^n)$.

Condition 2. *The function $f(t,x)$ has the uniformly continuous partial derivative $f_x'(t,x)$ in $G^0(\rho)$.*

We set $\mathbb{X} = \{v[t] = \left(v_1(t), v_2(t), \dots, v_N(t)\right) \in \tilde{\mathbb{C}}([0,T], \Delta_N, \mathbb{R}^{nN}) \colon\ v_r(t_{r-1}) = 0,\ r = \overline{1,N}\}$, $\mathbb{Y} = \tilde{\mathbb{C}}([0,T], \Delta_N, \mathbb{R}^{nN})$, and introduce the linear operator $H \colon \mathbb{X} \to \mathbb{Y}$ in the following way:

$$Hv[t] = w^{(1)}[t],$$

where

$$w^{(1)}[t] = \left(w_1^{(1)}(t), w_2^{(1)}(t), \dots, w_N^{(1)}(t)\right),$$

$$w_r^{(1)}(t) = \dot{v}_r(t) - \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int\limits_{t_{j-1}}^{t_j} \psi_k(\tau) v_j(\tau) d\tau, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1,N}$$

The domain of H is $D(H) = \{v[t] = \left(v_1(t), v_2(t), \dots, v_N(t)\right) \in \mathbb{X}$, where $v_r(t)$ is continuously differentiable on $[t_{r-1}, t_r]$, $r = \overline{1,N}\}$.

We can now rewrite the special Cauchy problem (4), (5) with $\lambda = \hat{\lambda} \in S\left(\lambda^{(0)}, \rho_\lambda\right)$ in the form of the nonlinear operator equation

$$Hv[t] + F\left(v[t], \hat{\lambda}\right) = 0 \tag{6}$$

with

$$F\left(v[t], \hat{\lambda}\right) = \left(w_1^{(2)}\left(t, \hat{\lambda}\right), w_2^{(2)}\left(t, \hat{\lambda}\right), \dots, w_N^{(2)}\left(t, \hat{\lambda}\right)\right),$$

$$w_r^{(2)}(t,\hat{\lambda}) = -f(t, v_r(t)+\hat{\lambda}_r) - \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau) d\tau \hat{\lambda}_j,$$

$$t \in [t_{r-1}, t_r], \quad r = \overline{1,N}.$$

Condition 2 guarenties the existence and uniform continuity of the Frechet derivative $F'_v(v[t],\hat{\lambda})$ in $S(v^{(0)}[t],\rho_u)$, which can be written in the form:

$$F'_v(v[t],\hat{\lambda}) = diag\left\{-\frac{\partial f(t, v_1(t)+\hat{\lambda}_1)}{\partial x}, \dots, -\frac{\partial f(t, v_N(t)+\hat{\lambda}_N)}{\partial x}\right\}.$$

Let $\mathbb{L}(\mathbb{Y},\mathbb{X})$ be the space of linear bounded operators: $\Lambda: \mathbb{Y} \to \mathbb{X}$ with the induced norm.

Fix $\hat{\lambda} \in S(\lambda^{(0)},\rho_\lambda)$, $\hat{v}^{(0)}[t] \in S(v^{(0)}[t],\rho_u) \cap D(H)$, and $\hat{\rho}_u > 0$.

Theorem 2. *Let the following conditions be fulfilled:*

(i) $F'_v(v[t],\hat{\lambda})$ *is uniformly continuous in* $S(\hat{v}^{(0)}[t],\hat{\rho}_u)$*;*

(ii) *the operator* $H + F'_v(v[t],\hat{\lambda}): \mathbb{X} \to \mathbb{Y}$ *has a bounded inverse and* $\|[H + F'_v(v[t],\hat{\lambda})]^{-1}\|_{\mathbb{L}(\mathbb{Y},\mathbb{X})} \le \hat{\chi}$ *for all* $v[t] \in S(\hat{v}^{(0)}[t],\hat{\rho}_u)$, $\hat{\chi}$ *is constant;*

(iii) $\hat{\chi}\|H\hat{v}^{(0)}[\cdot] + F(\hat{v}^{(0)}[\cdot],\hat{\lambda})\|_3 < \hat{\rho}_u$.

Then there exist numbers $\alpha_k \ge 1$, $k = 0,1,2,\dots$, *such that the sequence* $\{\hat{v}^{(k)}[t]\}$, *generated by the iterative process*

$$\hat{v}^{(k+1)}[t] = \hat{v}^{(k)}[t] - \frac{1}{\alpha_k}\left[H + F'_v(\hat{v}^{(k)}[t],\hat{\lambda})\right]^{-1} \times$$

$$\times \left[H\hat{v}^{(k)}[t] + F(\hat{v}^{(k)}[t],\hat{\lambda})\right], \quad k = 0,1,2,\dots,$$

converges to $v[t,\hat{\lambda}]$, *an isolated solution to equation (6) in* $S(\hat{v}^{(0)}[t],\hat{\rho}_u)$, *and the following estimate holds:*

$$\|v[\cdot,\hat{\lambda}] - \hat{v}^{(0)}[\cdot]\|_3 \le \chi\|H\hat{v}^{(0)}[\cdot] + F(\hat{v}^{(0)}[\cdot],\hat{\lambda})\|_3.$$

Consider the special Cauchy problem for the system of linear Fredholm integro-differential equations with parameters

$$\frac{d\vartheta_r}{dt} = f'_x(t, v_r(t)+\hat{\lambda}_r)\vartheta_r +$$

$$+\sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)\vartheta_j(\tau) d\tau + g_r(t), \quad t \in [t_{r-1}, t_r], \tag{7}$$

$$\vartheta_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \tag{8}$$

Theorem 1 and the interrelation between the special Cauchy problem (4), (5) and operator equation (6) yield the following assertion.

Theorem 3. *Let condition 2 be fulfilled, the special Cauchy problem (7), (8) be well-posed with constant* $\hat{\chi}$ *for all* $v[t] \in S(\hat{v}^{(0)}[t], \hat{\rho}_u)$*, and the following inequality be valid:*

$$\hat{\chi} \max_{r=\overline{1,N}} \max_{t\in[t_{r-1},t_r]} \Bigg\| \dot{\hat{v}}_r^{(0)}(t) - f\left(t, \hat{v}_r^{(0)}(t) + \hat{\lambda}_r\right) - \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)\left[\hat{v}_j^{(0)}(\tau) + \hat{\lambda}_j\right] d\tau \Bigg\| < \hat{\rho}_u.$$

Then there exist numbers $\alpha_k \geq 1$, $k = 0, 1, 2, \ldots$, *such that the sequence* $\{\hat{v}^{(k)}[t]\}$*, generated by the iterative process*

$$\hat{v}^{(k+1)}[t] = \hat{v}^{(k)}[t] + \Delta v^{(k)}[t, \hat{\lambda}], \quad k = 0, 1, 2, \ldots,$$

where $\Delta v^{(k)}[t, \hat{\lambda}]$ *is the solution to the special Cauchy problem for the system of linear integro-differential equations with parameters*

$$\frac{d\Delta v_r}{dt} = f'_x\left(t, v_r^{(k)}(t, \hat{\lambda}_1, \ldots, \hat{\lambda}_N) + \hat{\lambda}_r\right)\Delta v_r + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)\Delta v_j(\tau) d\tau - \frac{1}{\alpha_k}\Big\{\dot{v}_r^{(k)}(t, \hat{\lambda}_1, \ldots, \hat{\lambda}_N) - \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)\left[v_j^{(k)}(\tau, \hat{\lambda}_1, \ldots, \hat{\lambda}_N) + \hat{\lambda}_j\right] d\tau - f\left(t, v_r^{(k)}(t, \hat{\lambda}_1, \ldots, \hat{\lambda}_N) + \hat{\lambda}_r\right)\Big\}, \quad t \in [t_{r-1}, t_r],$$

$$\Delta v_r(t_{r-1}, \hat{\lambda}_1, \ldots, \hat{\lambda}_N) = 0, \quad r = \overline{1, N},$$

converges to $v[t, \hat{\lambda}]$*, an isolated solution to problem (4), (5) in* $S(\hat{v}^{(0)}[t], \hat{\rho}_u)$*, and*

$$\left\|v[\cdot, \hat{\lambda}] - \hat{v}^{(0)}[\cdot]\right\|_3 \leq \hat{\chi} \max_{r=\overline{1,N}} \max_{t\in[t_{r-1},t_r]} \Bigg\| \dot{\hat{v}}_r^{(0)}(t) - f\left(t, \hat{v}_r^{(0)}(t) + \hat{\lambda}_r\right) - \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)\left[\hat{v}_j^{(0)}(\tau) + \hat{\lambda}_j\right] d\tau \Bigg\|.$$

In the second Section, by using the solution to the special Cauchy problem (4), (5), a general solution to equation (1) is constructed, and its properties are established. Taking into account the relationship between the special Cauchy problems (2), (3) and (4), (5), we give the following definition.

Definition 1. *Let the function system* $v[t,\lambda] = \big(v_1(t,\lambda), v_2(t,\lambda), \dots, v_N(t,\lambda)\big) \in S(0,\rho_v)$ *be a solution to the special Cauchy problem (4), (5) with the parameter* $(\lambda_1,\lambda_2,\dots,\lambda_N) \in S\big(\lambda^{(0)},\rho_\lambda\big)$. *Then the function* $x(\Delta_N,t,\lambda)$, *given by the equalities* $x(\Delta_N,t,\lambda) = \lambda_r + v_r(t,\lambda)$ *for* $t \in [t_{r-1},t_r)$, $r=\overline{1,N}$, *and* $x(\Delta_N,T,\lambda) = \lambda_N + v_N(T,\lambda)$, *is called a* Δ_N *general solution to equation (1) in* $G^0(\Delta_N,\rho)$.

Theorem 4. *Let a piecewise continuous on* $[0,T]$ *function* $\tilde{x}(t)$ *with the possible discontinuity points* $t=t_p$, $p=\overline{1,N-1}$, *be given, and* $\big(t,\tilde{x}(t)\big) \in G^0(\Delta_N,\rho)$. *Assume that the function* $\tilde{x}(t)$ *has a continuous derivative and satisfies equation (1) for all* $t \in (0,T)\backslash\big\{t_p, p=\overline{1,N-1}\big\}$. *Then there exists a unique* $\tilde{\lambda} = \big(\tilde{\lambda}_1,\tilde{\lambda}_2,\dots,\tilde{\lambda}_N\big) \in S\big(\lambda^{(0)},\rho_\lambda\big)$, *such that the equality* $(\Delta_N,t,\lambda^*) = x^*(t)$ *holds for all* $t \in [0,T]$.

Corollary 1. *Let* $x^*(t)$ *be a solution to equation (1), and* $\big(t,x^*(t)\big) \in G^0(\Delta_N,\rho)$. *Then there exists a unique* $\lambda^* = (\lambda_1^*,\lambda_2^*,\dots,\lambda_N^*) \in S\big(\lambda^{(0)},\rho_\lambda\big)$, *such that the equality* $x(\Delta_N,t,\lambda^*) = x^*(t)$ *holds for all* $t \in [0,T]$.

The nonlinear boundary value problem for the Fredholm integro-differential equation (1) with the boundary condition

$$g[x(0),x(T)] = 0 \qquad (10)$$

is considered.

The Δ_N general solution to equation (1) allows us to reduce the solvability of boundary value problem (1), (10) to the solvability of the system of nonlinear algebraic equations in parameters

$$Q_*(\Delta_N;\lambda) = 0, \quad \lambda \in \mathbb{R}^{nN}. \qquad (11)$$

Theorem 5. *Let the function* $x^*(t)$ *be a solution to problem (1), (10) and* $\big(t,x^*(t)\big) \in G^0(\Delta_N,\rho)$. *Then the vector* $\lambda^* = (\lambda_1^*,\lambda_2^*,\dots,\lambda_N^*) \in S\big(\lambda^{(0)},\rho_\lambda\big)$ *with elements* $\lambda_r^* = x^*(t_{r-1})$, $r=\overline{1,N}$, *is a solution to equation (11). Vice versa, if* $\tilde{\lambda} = \big(\tilde{\lambda}_1,\tilde{\lambda}_2,\dots,\tilde{\lambda}_N\big) \in S\big(\lambda^{(0)},\rho_\lambda\big)$ *is a solution to equation (11), then the function* $\tilde{x}(t) = x(\Delta_N,t,\lambda)$ *is a solution to problem (1), (10) and* $\big(t,\tilde{x}(t)\big) \in G^0(\Delta_N,\rho)$.

To solve the system of nonlinear algebraic equations (11), we use the following statement.

Theorem 6. *Let the following conditions be fulfilled:*

(i) *the Jacobi matrix* $\frac{\partial Q_*(\Delta_N;\lambda)}{\partial\lambda}$ *is uniformly continuous in* $S\big(\lambda^{(0)},\rho_\lambda\big)$;

(ii) $\frac{\partial Q_*(\Delta_N;\lambda)}{\partial\lambda}$ *is invertible and* $\left\|\left[\frac{\partial Q_*(\Delta_N;\lambda)}{\partial\lambda}\right]^{-1}\right\| \leq \gamma^*$ *for all* $\lambda \in S(\lambda^{(0)}, \rho_\lambda)$, γ^* *constant;*

(iii) $\gamma^*\|Q_*(\Delta_N;\lambda^{(0)})\| < \rho_\lambda$.

Then there exists $\alpha_0 \geq 1$ *such that for any* $\alpha \geq \alpha_0$ *the sequence* $\alpha \geq \alpha_0$ *generated by the iterative process*

$$\lambda^{(k+1)} = \lambda^{(k)} - \frac{1}{\alpha}\left[\frac{\partial Q_*(\Delta_N;\lambda^{(k)})}{\partial\lambda}\right]^{-1} Q_*(\Delta_N;\lambda^{(k)}), \quad k = 0, 1, 2, \ldots,$$

converges to λ^**, an isolated solution to equation (11) in* $S(\lambda^{(0)}, \rho_\lambda)$ *and*

$$\|\lambda^* - \lambda^{(0)}\| \leq \gamma^*\|Q_*(\Delta_N;\lambda^{(0)})\|.$$

This section also discusses the quasilinear Fredholm integro-differential equation

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^{m} \varphi_k(t)\int_0^T \psi_k(\tau)x(\tau)d\tau + f_0(t) + \varepsilon f(t,x), \qquad (12)$$

Applying the parametrization method to equation (12) for the Δ_N partition, we obtain the special Cauchy problem with parameters of the form

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + \sum_{k=1}^{m} \varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} \psi_k(\tau)[u_j(\tau) + \lambda_j]d\tau + f_0(t) +$$

$$+\varepsilon f(t, u_r + \lambda_r), \quad t \in [t_{r-1}, t_r), \qquad (13)$$

$$u_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \qquad (14)$$

Let $y(\Delta_N, t, \lambda)$ be the Δ_N general solution to the linear integro-differential equation

$$\frac{dy}{dt} = A(t)y + \sum_{k=1}^{m} \varphi_k(t)\int_0^T \psi_k(\tau)y(\tau)d\tau + f_0(t), \quad t \in [0,T], \quad y \in \mathbb{R}^n. \qquad (15)$$

Equation (15) is reduced to the special Cauchy problem

$$\frac{dv_r}{dt} = A(t)(v_r + \lambda_r) + \sum_{k=1}^{m} \varphi_k(t)\sum_{j=1}^{N}\int_{t_{j-1}}^{t_j} \psi_k(\tau)[v_j(\tau) + \lambda_j]d\tau +$$

$$+f_0(t), \qquad t \in [t_{r-1}, t_r), \qquad (16)$$

$$u_r(t_{r-1}) = 0, \quad r = \overline{1,N}. \qquad 17)$$

Given a vector $\lambda^{(0)} = \left(\lambda_1^{(0)}, \lambda_2^{(0)}, \dots, \lambda_N^{(0)}\right) \in \mathbb{R}^{nN}$ and numbers $\rho_\lambda > 0$, $\rho > \rho_\lambda$, $\rho_v = \rho - \rho_\lambda$, we choose the piecewise continuous on $[0, T]$ function $y^{(0)}(t) = y\left(\Delta_N, t, \lambda^{(0)}\right)$ the function system $v^{(0)}[t] = \left(v_1^{(0)}(t), v_2^{(0)}(t), \dots, v_N^{(0)}(t)\right)$ with elements $v_r^{(0)}(t) = y^{(0)}(t) - \lambda_r^{(0)}$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$, and compose the following sets:

$$G^0(\rho) = \left\{(t, x): t \in [0, T], \left\|x - y^{(0)}(t)\right\| < \rho\right\},$$

$$S\left(\lambda^{(0)}, \rho_\lambda\right) = \left\{\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N) \in \mathbb{R}^{nN}: \left\|\lambda_r - \lambda_r^{(0)}\right\| < \rho_\lambda, \quad r = \overline{1, N}\right\},$$

$$S\left(v^{(0)}[t], \rho_v\right) = \left\{u[t] \in \mathbb{C}([0, T], \Delta_N, \mathbb{R}^{nN}): \left\|u[\cdot] - v^{(0)}[\cdot]\right\|_2 < \rho_v\right\},$$

$$G_p^0(\rho) = \left\{(t, x): t \in \left[t_{p-1}, t_p\right), \left\|x - y^{(0)}(t)\right\| < \rho\right\}, \quad p = \overline{1, N-1},$$

$G_N^0(\rho) = \left\{(t, x): t \in [t_{N-1}, t_N], \left\|x - y^{(0)}(t)\right\| < \rho\right\}$, and

$$G^0(\Delta_N, \rho) = \bigcup_{r=1}^{N} G_r^0(\rho).$$

We represent problem (13), (14) as an operator equation and apply an iterative process for finding its solution. We introduce a linear operator $H: \mathbb{X} \to \mathbb{Y}$ in the following way:

$$Hu[t] = \left(w_1^{(1)}(t), w_2^{(1)}(t), \dots, w_N^{(1)}(t)\right),$$

where

$$w_r^{(1)}(t) = \dot{u}_r(t) - A(t)u_r - \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau) u_j(\tau) d\tau,$$

$$t \in [t_{r-1}, t_r), \qquad r = \overline{1, N}.$$

We can now write down the special Cauchy problem (13), (14) in the form of the nonlinear operator equation

$$Hu[t] = \varepsilon F(u[t], \lambda) + F_0[t, \lambda],$$

where

$$F(u[t], \lambda) = \left(w_1^{(2)}(t), w_2^{(2)}(t), \dots, w_N^{(2)}(t)\right),$$

$$w_r^{(2)}(t) = f(t, u_r(t) + \lambda_r), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N}.$$

Theorem 7. *Let the special Cauchy problem (16), (17) be well-posed with a constant χ and the following inequalities be valid:*

(i) $\|f(t, x') - f(t, x'')\| \leq L \parallel x' - x'' \parallel$, *$L$ is constant.*, $(t, x'), (t, x'') \in G^0(\rho)$;

(ii) $q_\varepsilon = \varepsilon \chi L < 1$;

(iii) $\frac{1}{1-q_\varepsilon}\varepsilon\chi \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \|f(t,v_r(t,\lambda)+\lambda_r)\| < \rho_v$ *for all* $\lambda \in S(\lambda^{(0)},\rho_\lambda)$.

Then for any $\lambda \in S(\lambda^{(0)},\rho_\lambda)$ *there exists a unique function system* $u[t,\lambda,\varepsilon] = (u_1(t,\lambda,\varepsilon),u_2(t,\lambda,\varepsilon),\dots,u_N(t,\lambda,\varepsilon))$, *the solution to the special Cauchy problem (13), (14) belonging to* $S(v^{(0)}[t],\rho_v)$ *and the following inequality is true:*

$$\|u[\cdot,\lambda,\varepsilon]-v[\cdot,\lambda]\|_2 \le \frac{1}{1-q_\varepsilon}\varepsilon\chi \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \|f(t,v_r(t,\lambda)+\lambda_r)\|.$$

Definition 2. Let a function system $u[t,\lambda,\varepsilon] = (u_1(t,\lambda,\varepsilon),u_2(t,\lambda,\varepsilon),\dots,u_N(t,\lambda,\varepsilon)) \in S(v[t,\lambda],\rho_v)$ be the unique solution to the special Cauchy problem *(13)*, *(14)* with the parameter $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_N) \in S(\lambda^{(0)},\rho_\lambda)$. Then the function $x(\Delta_N,t,\lambda,\varepsilon)$, given by the equalities

$$x(\Delta_N,t,\lambda,\varepsilon)=\lambda_r+u_r(t,\lambda,\varepsilon) \text{ for } t\in[t_{r-1},t_r),\ r=\overline{1,N}, \text{ and}$$

$$x(\Delta_N,T,\lambda,\varepsilon)=\lambda_N+\lim_{t\to T-0}u_N(t,\lambda,\varepsilon)$$

is said to be a Δ_N general solution to equation (12) in $G^0(\Delta_N,\rho)$.

Definition 2 and theorem 7 imply the following assertion.

Theorem 8. *Under conditions of Theorem 7, there exists a function* $x(\Delta_N,t,\lambda,\varepsilon)$, *which is a unique* Δ_N *general solution to equation (12) in* $G^0(\Delta_N,\rho)$, *and this function can be represented in the form*

$$x(\Delta_N,t,\lambda,\varepsilon)=y(\Delta_N,t,\lambda)+\Delta x(\Delta_N,t,\lambda,\varepsilon),$$

where the function $\Delta x(\Delta_N,t,\lambda,\varepsilon)$ *is compiled by the equalities*

$$\Delta x(\Delta_N,t,\lambda,\varepsilon)=u_r(t,\lambda,\varepsilon)-v_r(t,\lambda), \text{ for } t\in[t_{r-1},t_r),\ r=\overline{1,N},$$

$$\Delta x(\Delta_N,T,\lambda,\varepsilon)=\lim_{t\to T-0}u_N(t,\lambda,\varepsilon)-\lim_{t\to T-0}v_N(t,\lambda).$$

Moreover, the following estimate is valid:

$$\sup_{t\in[0,T)}\|\Delta x(\Delta_N,t,\lambda,\varepsilon)\| \le \frac{1}{1-q_\varepsilon}\varepsilon\chi \max_{r=\overline{1,N}} \sup_{t\in[t_{r-1},t_r)} \|f(t,v_r(t,\lambda)+\lambda_r)\|.$$

In Subsection 2.4, we study the solvability of the quasilinear boundary value problem for the Fredholm integro-differential equation

$$\frac{dx}{dt}=A(t)x+\sum_{k=1}^{m}\varphi_k(t)\int_0^T\psi_k(\tau)x(\tau)d\tau+f_0(t)+\varepsilon f(t,x), \quad t\in[0,T], \quad x\in\mathbb{R}^n,$$

$$Bx(0)+Cx(T)=d, \quad d\in\mathbb{R}^n.$$

Section 3 is devoted to the development of an algorithm for finding a solution to the nonlinear boundary value problem for the integro-differential equation.

In Subsection 3.1, we consider the special Cauchy problem for the systems of nonlinear integro-differential equations

$$\frac{du_r}{dt} = f_0(t, u_r + \lambda_r) + \sum_{j=1}^{N} \int_{(j-1)h}^{jh} f_1\big(t, s, u_j(s) + \lambda_j\big)ds, \quad t \in [(r-1)h, rh), \quad (18)$$

$$u_r[(r-1)h] = 0, \quad r = \overline{1, N}, \quad (19)$$

which arises while applying the parameterization method to the system of nonlinear Fredholm integro-differential equations

$$\frac{dx}{dt} = f_0(t, x) + \int_0^T f_1\big(t, s, x(s)\big)ds, \quad t \in [0, T], \quad x \in \mathbb{R}^n.$$

An algorithm for finding a numerical solution to the problem (18), (19) is developed.

This subsection also offers an algorithm for finding a numerical solution to the special Cauchy problem for the systems of nonlinear integro-differential equations with nonlinear differential part

$$\frac{dv_r}{dt} = f(t, v_r + \lambda_r) + \sum_{k=1}^{m} \varphi_k(t) \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \psi_k(\tau)[v_j(\tau) + \lambda_j]d\tau, \quad t \in [t_{r-1}, t_r],$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1, N}.$$

Theorem 2 provides the convergence of the proposed algorithm.

In Subsection 3.2, an algorithm for finding a solution to the nonlinear boundary value problem for the Fredholm integro-differential equation (1), (10) is developed.

In Subsection 3.3, based on the parameter continuation method , one approach to solving the problem of choosing the initial guess solutions to the special Cauchy problem (4), (5) and systems of nonlinear algebraic equations (11) is proposed.

Since

$$\int_0^t \varphi\big(t, s, x(s)\big)ds = \int_0^T \tilde{\varphi}\big(t, s, x(s)\big)ds,$$

where

$$\tilde{\varphi}\big(t, s, x(s)\big) = \begin{cases} \varphi\big(t, s, x(s)\big), & a \le s \le t, \\ 0, & t < s \le b, \end{cases}$$

then the Volterra integro-differential equation

$$\dot{x} = \varepsilon X\left(t, x, \int_0^t \varphi\big(t, s, x(s)\big)ds\right)$$

is a particular case of the Fredholm integro-differential equation

$$\dot{x} = \varepsilon X\left(t, x, \int_0^T \tilde{\varphi}\big(t, s, x(s)\big)ds\right).$$

In this connection, in Subsection 3.4, the averaging method is applied to the study of the existence of solutions to the boundary value problems for the systems of Volterra integro-differential equations. It is shown that if the average boundary value problem has a solution, then the original problem also has a solution. It is important that the average for a system of integro-differential equations is a simpler system of ordinary differential equations.

Thus, the dissertation work investigates the solvability of nonlinear boundary value problems for the integro-differential equations and offers following new scientific results:

- sufficient conditions for the existence of solutions to the special Cauchy problem for the systems of nonlinear integro-differential equations with parameters are obtained;
- iterative methods for solving the special Cauchy problem for the systems of nonlinear integro-differential equations with parameters and their numericalimplementations are proposed;\
- the Δ_N general solution to the Fredholm integro-differential equation with nonlinear differential part is constructed and its properties are established;
- the parameterization method is extended for solving the nonlinear boundary value problem for the Fredholm integro-differential equation;
- algorithms for solving the nonlinear boundary value problems for the Fredholm integro-differential equations and their numerical implementations are developed;
- sufficient conditions for the existence of an isolated solution to a nonlinear boundary value problem for the Fredholm integro-differential equation are established;
- a system of nonlinear algebraic equations in parameters for the boundary value problem for the Fredholm integro-differential equation with nonlinear differential part is constructed and an algorithm for finding its solution is proposed;
- algorithms for finding initial guess solutions to the nonlinear special Cauchy problem and constructed system of nonlinear algebraic equations are developed;
- the averaging method is applied to the study of the existence of solutions to the initial and boundary value problems for the nonlinear Volterra integro-differential equation.