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THE LYAPUNOV METHOD IN THE MULTIPERIODIC OSCILLATIONS IN SYSTEMS WITH AN OPERATOR OF DIFFERENTIATION WITH RESPECT TO DIRECTIONS OF A VECTOR FIELD

ABSTRACT
of the thesis for the degree of Doctor of Philosophy (PhD) in the specialty 6D060100 – Mathematics

Relevance of the topic. The urgent need for such a study can be justified by two applications, which can be conditionally called mechanical and theoretical. As is known, the partial differential equations can be interpreted as mathematical models of a flow of noninteracting particles in a direction of field in many cases. Then the process is described by the partial differential equation as the motion of a continuous medium, and by a system of ordinary differential equations as the motion of particles.

Consequently, the study under consideration has a *mechanical application* but, on the other hand, its connection with ordinary equations can be used to solve the problems of the theory of ordinary differential equations. Thus we have a *theoretical application* of this study.

The work, both on these applications and in content, refers to the study of problems in the theory of oscillations, in which the scientific schools of N.N.Bogoliubov - Yu.A. Mitropolsky - A.M. Samoilenko and A.N. Kolmogorov - V.I. Arnold on KAM theory have a prominent position. But it should be noted that here the problems are studied in the terms of partial differential equations based on the research of V.Kh. Kharasakhal - D.U. Umbetzhanov.

A time is multidimensional in this study. This is due to the rational incommensurability of its components. Obviously, the velocity of movements is determined by the differentiation operator on directions of a given vector field of time and space variables. In this ways, the partial differential equations considered in the thesis have the form of matrix-vector equations with the differentiation operator. Many natural processes such as the propagation of sound, light, heat and electromagnetic waves in a medium are well-known examples described by such equations. For example, the Euler equations from hydromechanics, the Hopf equation from the dynamics of shock waves, and the eikonal equation from optics can be described in the form of equations with the differentiation operator on directions of corresponding vector fields. These processes have a wave nature. The studies of the problems of multiperiodic (oscillating) solutions of their mathematical models have the most prominent in the theory of wave processes. The relevance of the topic of this direction from various points of view is also noted in the fundamental works of M.A.Lyapunov, A. Poincaré, A.A. Andronov, V.V. Niemytzki, V.V. Stepanov, W.Hamilton, C. Jacobi,, N.N. Bogoliubov, A.N. Kolmogorov, Yu.A. Mitropolsky, V.I. Arnold, J. Moser, K.L. Siegel, V.I. Zubov, R. Courant, B.L. Rozhdestvensky, N.N. Yanenko, G. Whitham, S. Farlow, N.V. Karlov, N.A. Kirichenko, P. Bol, G.Bor, B.M. Levitan, I.G. Petrowsky, L.S. Pontryagin, M.A. Krasnoselsky, M.G.Krein, V. Vazov, A.M. Samoilenko, M. Urabe, Y. Sibuja, I. G. Malkin, N.P.Erugin, B.P. Demidovic, V.A. Yakubovich, V.M. Starzhinsky, V.Kh.Kharasakhal, D.U. Umbetzhanov.

Thus, the relevance of the thesis research is justified by its *possible applied aspect in the mechanics, the physics and in the scientific and technical process*.

There is a need to move from problems of the ordinary equations to the equivalent problems for the partial differential equations in order to justify the theoretical applications of the thesis research. This is not an easy question; it depends on the skill of the researcher. For example, C. Jacobi put forward a research method with a transition to the first-order partial differential equation when investigated the question of integrating Hamilton's canonical systems. In this ways, Jacobi methods appeared for the study of the Hamilton-Jacobi equations.

Another important example is the transition from problems of quasiperiodic solutions of the systems of ordinary differential equations to the problems of multiperiodic solutions for the systems of partial differential equations of the first order with the differentiation operator, which is called the method of V.Kh.Kharasakhal and developed by D. U. Umbetzhanov, Zh. A. Sartabanov, A.B.Berzhanov and their followers. Systems and the research problems in the thesis relate to more general systems and problems for them. Consequently, the thesis research has a very important theoretical application in the theory of quasiperiodic solutions of systems of ordinary equations. It should be noted, that all the studies conducted here have this tendency. Thus, we can conclude that the thesis research relates to a relevancy problems.

The purpose of the thesis research. The investigation of multiperiodic oscillations based on the generalization of Lyapunov's method in the systems with the differentiation operator along the directions of the vector field, introduced by V.Kh.Kharasakhal in time variables and determined by various types of Lyapunov's system with respect to space variables.

Tasks of the thesis research.

a) Establishment of sufficient conditions for the existence of multiperiodic solutions of quasilinear systems of partial differential equations presented with the differentiation operator on directions of vector fields, introduced by V.Kh.Kharasakhal in time variables and determined by various types of Lyapunov's system with respect to space variables.

b) Construction of the integral representations of multiperiodic solutions of linear systems, both in noncritical and critical cases.

c) Investigation of multiperiodic oscillations of autonomous linear and quasilinear systems with the above differentiation operators, consideration of multiperiodically perturbed cases.

d) Investigation of multiperiodic oscillations in quasilinear systems with exponential-hyperbolic varying small parameter.

Object of research is the multiperiodic solutions of linear and quasilinear systems with the differentiation operator along the directions of the vector field introduced by V.Kh. Kharasakhal in time variables and determined by various types of Lyapunov's system with respect to space variables.

Research method. The methods and results of the theories of ordinary differential equations, partial differential equations of the first order, operator equations, generalized Lyapunov's method for studying the periodic motions, elements of the method KAM theory, method of successive approximations, Green's function method, methods of investigation of multiperiodic solutions of the systems of differential equations in non-critical cases are used in the dissertation work. Methods for investigating the multiperiodic solutions in critical and autonomous cases and method of a small parameter that varies according to an exponential-hyperbolic law have been developed on the basis of these methods.

Scientific novelty of the research.

a) Sufficient conditions for the existence of multiperiodic solutions of the linear systems with different differentiation operators, which differ in the forms of Lyapunov's system, are established.

b) Conditions for the existence of multiperiodic solutions of the linear systems in the critical case are obtained. A problem of this type has been studied for the first time.

c) Integral representations of multiperiodic solutions of the linear inhomogeneous systems are determined, which are suitable for the general case both in the critical and non-critical cases.

d) Sufficient conditions for the existence of multiperiodic solutions of the quasilinear systems are established for the noncritical case.

e) Autonomous linear and quasilinear systems are investigated and similar results are obtained for such systems.

f) Sufficient conditions for the existence of a multiperiodic solution of the one quasilinear system with an exponential-hyperbolic varying small parameter are indicated and its construction is substantiated by the method of successive approximations.

g) The well-known Lyapunov methods for periodic solutions of the ordinary equations for the multiperiodic case are generalized. Methods for investigating the problem of establishing the multiperiodic solutions have been developed and carried out strictly according to the following scheme: for the zeros of the differentiation operator; then for a homogeneous linear system; further, for an inhomogeneous linear system; and then, for a quasilinear system, their interrelationships were established by the initial data or by a small parameter.

The following results are submitted for defense.

- Necessary and sufficient conditions for the existence of multiperiodic solutions of linear systems with the differentiation operator along the directions of a vector field.

- Existence conditions and methods of investigation of a real-analytic multiperiodic solution of linear inhomogeneous systems in the critical case.

- Existence conditions and research methods for the multiperiodic solutions of autonomous linear systems.

- Integral representation for multiperiodic solutions of the linear inhomogeneous systems, which is corresponding both in critical and non-critical cases.

- Multiperiodic structural analysis of solutions of the linear systems with the differentiation operator on directions of a vector field.

- Existence conditions for multiperiodic solutions of the considered quasilinear systems in the noncritical case.

- Existence conditions and methods for studying a multiperiodic solution of one quasilinear system in the critical case with an exponential-hyperbolic varying parameter, in which the nonlinear part is a homogeneous form with respect to the required function.

The theoretical and practical value of the results. The results of the thesis are theoretical. An essential addition to the development of the theory of multiperiodic solutions of systems of equations of the first order are the obtained conditions for the existence of multiperiodic solutions and the proposed their integral representations; the research of multiperiodic solutions in the critical case, study of this question in the autonomous case of systems; the applicability of the method of successive approximations, when the nonlinear term of the system is a homogeneous form with respect to an unknown function with an exponential-hyperbolic varying small parameter.

These received innovations can be the impetus for the further development of the theory and can serve as the basis for the development of elective courses for students, postgraduate and doctoral students in physics, mathematics and engineering.

Personal contribution of the PhD student. All the results presented in the thesis obtained by the author. The participation of co-authors and scientific consultants consists in setting tasks and discussing the obtained results.

Approbation of the thesis results. The main results of the work were reported and discussed at the following events:

- International scientific-practical conference "Taimanov readings - 2017" (Uralsk, Kazakhstan, October 25, 2017);
- International scientific conference "The Problems of Applied Mathematics and Computer Science" (Aktobe, Kazakhstan, November 10-11, 2017);
- Scientific conference of Mathematicians of Kazakhstan "Actual problems of mathematics" (Turkestan, Kazakhstan, April 28-30, 2018);
- Fourth International Conference on Analysis and Applied Mathematics (ICAAM 2018) (Lefkosa (Nicosia), Turkey, September 6-9, 2018);
- The traditional international April mathematical conference. Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan (Almaty, Kazakhstan, April 5-8, 2021);
- Scientific seminar "Research of problems of nonlinear optimization of the systems with distributed parameters", Kyrgyz-Russian Slavic University, Bishkek, Kyrgyz Republic (seminar leader Doctor of Physical and Mathematical Sciences, professor A. Kerimbekov);

- Scientific seminar "Qualitative theory of the differential equations", Kyrgyz National University named after Zh. Balasagyn, Bishkek, Kyrgyz Republic (seminar leaders Doctor of Physical and Mathematical Sciences, professors A.Saadabaev, B.K Temirov);
- Scientific seminar "Qualitative and approximate methods of research of the differential equations", Institute of Mathematics and Mathematical Modeling of the Ministry of Education and Science of the Republic of Kazakhstan, Almaty (seminar leader Doctor of Physical and Mathematical Sciences, professor A.T. Assanova);
- Scientific seminar "The Problems of Applied Mathematics and Computer Science", Department of Mathematics, K. Zhubanov Aktobe Regional University, Aktobe, Kazakhstan (seminar leader Doctor of Physical and Mathematical Sciences, professor Zh. Sartabanov).

Publications. The results of the thesis were published in 11 papers, of which 1 article in rating scientific journals indexed in the Scopus database, 4 articles in publications recommended by the Committee for Control in Education and Science of the Ministry of Education and Science of the Republic of Kazakhstan, 1 article in the scientific journal of the Republic of Kazakhstan, the rest in the materials of international conferences, including 1 publication in the materials of foreign conferences (Scopus).

The volume and structure of the thesis. The dissertation consists of an introduction, two section (the first section includes 6 subsections, the second includes 3 subsections.), a conclusion and a list of references, including 110 titles, and appendices A and B.

The obtained results on the linear systems with the differentiation operator on directions of the vector field are highlighted **in the first section** of the work. A vector field consists of the sum of vector fields with respect to time (τ, t) and space $\zeta = (\xi, \eta)$ variables. The field with respect to time variables is known from the research of V.Kh. Kharasakhal, where differentiation is carried out along the directions of the main diagonal of these variables, and the field in space variables is determined by Lyapunov's system. Obviously, the elements of the operator, the matrix of coefficients and the free term of the system depend on both time and space variables, moreover, they are changed multiperiodically in time variables (τ, t) by period (θ, ω).

The problems of multiperiodic solutions with respect to time variables of the consideration systems are investigated in the dissertation work. The input data of the system depending on the types of studied problems were being modified in the process of research.

We present a general view of the system under study

$$Dx = P(\tau, t, \zeta)x + f(\tau, t, \zeta, x)$$

with the differentiation operator

$$D = \frac{\partial}{\partial \tau} + \left\langle e, \frac{\partial}{\partial \tau} \right\rangle + \left\langle M\zeta + \psi(\tau, t, \zeta), \frac{\partial}{\partial \zeta} \right\rangle$$

with respect to the required vector-function x for convenience, where the vector $e = (1,...,1)$ and the expression $M\zeta + \psi$ are defined the differentiation field, M is a constant matrix, P is a matrix, ψ and f are vector-functions, $\frac{\partial}{\partial t} = \left(\frac{\partial}{\partial t_1},...,\frac{\partial}{\partial t_m}\right)$ and $\frac{\partial}{\partial \zeta} = \left(\frac{\partial}{\partial \xi_1},\frac{\partial}{\partial \eta_1},...,\frac{\partial}{\partial \xi_l},\frac{\partial}{\partial \eta_l}\right)$ are vectors, $\langle \ , \ \rangle$ denotes the scalar product.

We will consider special cases of D_j, $j = \overline{1,5}$ for the operator D in the future.

Special case is considered in subsections 1.1 and 1.2, when the system consists of two equations, and they differ from each other by the coefficients for unknowns, i.e. matrix of coefficients constant in 1.1 and variable in 1.2, and the field of the differentiation operator D_1 with respect to space variables is determined by the linear Lyapunov's system of two variables.

The zeros of the operator D_1 are defined and their properties of multiperiodicity with respect to time variables are studied in the subsection 1.1. Solution of the initial value problem for the equation $D_1 x = Ax$ with a constant matrix A is constructed and the condition for its multiperiodicity is established, that the equation in the noncritical case has only a trivial multiperiodic solution is proved. In the critical case, sufficient conditions are obtained for the absence of nontrivial solutions; sufficient conditions are established for the existence of a unique multiperiodic solution of an inhomogeneous equation $D_1 x = Ax + f(\tau,t,\zeta)$. A similar study was carried out when the matrix of the coefficients of the system is constant on the diagonal.

In the subsection 1.2 sufficient conditions for the existence of ω-periodic with respect to t solutions of a homogeneous equations $D_1 x = P(\tau,t)x$ with (θ,ω)-periodic matrix of coefficients are established, a corollary of the theorem on conditions for the absence of nontrivial multiperiodic solution is given. Conditions for the existence of a unique multiperiodic solution of an inhomogeneous equation $D_1 x = P(\tau,t)x + f(\tau,t,\zeta)$ are established in the absence of a nontrivial multiperiodic solution of the corresponding homogeneous equation. Here the study is carried out in terms of Green's function and an estimate of this solution is found. Similar results were obtained for an equation with a matrix coefficient $P(\tau,t,\zeta)$. The concept of a singular solution for an autonomous inhomogeneous equation $D_1 x = P(\zeta)x + f(\zeta)$ is introduced and conditions for the periodicity of solutions are obtained. The existence of the unique periodic solution of the autonomous equation is proved.

The systems with the operator D_2 are investigated in the subsection 1.3, where the matrix M of the linear part of the Lyapunov's system has the form $M = 2\pi v^0 I_2$ with constant v^0 and the symplectic unit I_2 of the second order. The conditions for the existence of analytic in ζ, ω-periodic in t and quasiperiodic in τ with a definite frequency basis of the zeros of operator D_2 is established while investigating. Sufficient conditions for the existence and absence of nontrivial multiperiodic solutions of the homogeneous equations $D_2 x = Ax$ with some matrix $A = [a_{ij}]$, $a_{11} = a_{22} = \alpha$, $a_{12} = -a_{21} = \beta$ are obtained. Sufficient conditions for the existence of analytic in ζ, ω-periodic in t and quasiperiodic in τ solutions of an inhomogeneous autonomous equations $D_2 x = Ax + f(\zeta)$ are established. The conditions are determined which the given inhomogeneous equation can allow the solution that is analytic with respect to ζ and depends only on ζ, that also belongs to the number of multiperiodic in (τ, t) solutions. Also, the main problem for the inhomogeneous equation for case $f = f(\tau, t, \zeta)$ in terms of Green's function is investigated.

Subsection 1.4 is devoted to the investigation of the multiperiodic solutions of systems with the differentiation operator D_3 when, $M = J$, $\psi = \psi(\zeta)$, where J is a $(2l+2)$-dimensional block-diagonal matrix. 1) The dimension $m = l$ of the variable t and the vector period $(\theta, \omega_1, ..., \omega_l)$ are determined, 2) sufficient conditions for the existence of multiperiodic zeros of the differentiation operator D_3 are established based on the properties of solutions of Lyapunov's system with known the first integrals $H(\xi, \eta)$.

Further, in the noncritical case, sufficient conditions for the existence of unique multiperiodic solution of the autonomous inhomogeneous systems $D_3 x = Ax + f(\zeta)$ in terms of the Green's function are established. The results investigation of this system is generalized to the general autonomous case of the systems $D_3 x = P(\zeta)x + f(\zeta)$ under the assumption that the corresponding homogeneous system is not critical.

Also, the system with a common free term $f(\tau, t, \zeta)$ is investigated and sufficient conditions for the existence of a unique (θ, ω)-periodic solution are obtained.

It is important to consider equations with a differentiation operator, which is determined by the fields given by systems close to Lyapunov's system, different from them by oscillatory perturbations in time variables. In this regard, problems of the systems with the differentiation operator D_4 are investigated in the subsection 1.5, where $e = a(\tau, t)$, $M = J$, $\psi = g(\tau)$.

The properties of the solutions of equation defining the zeros of the operator D_4 are studied. Lemmas that reveal various properties of the matricant depending on the form of the eigenvalues of the constant matrix of coefficients of the homogeneous equations $D_4 x = Ax$ and the theorem on the structure of the matricant are proved; the multiperiodic structure of the general solution is determined; conditions for the existence of the multiperiodic solutions and for the absence of such solutions, except for the trivial one, are obtained; the class of (θ, ω)-periodic solutions corresponding to the same zeros of the operator D_4 is highlighted; sufficient conditions for the existence and uniqueness of the multiperiodic solution of inhomogeneous systems $D_4 x = Ax + f(\tau, t, \zeta)$ are established, where the free term can be considered perturbed

(θ, ω)-periodic in (τ, t) force. The multiperiodic structure of solutions of the initial-multiperiodic problems for the linear systems with operator D_4 is determined.

Subsection 1.6 is considered the system with the differentiation operator in time variables D_5, introduced by V.Kh. Kharasakhal, where $M = \psi = 0$. Conditions for the absence of multiperiodic solutions of homogeneous systems are obtained when the matrix of coefficients of the system has a pair of purely imaginary eigenvalues, and the rest of the eigenvalues have nonzero real parts. The Green function for the problem of a (θ, ω)-periodic solutions of the inhomogeneous systems $D_5 x = Ax + \varepsilon^0 f(\tau, t)$ is given, when the frequencies of free and forced oscillations satisfy the condition of strong incommensurability. As a result, the conditions for the existence of a unique (θ, ω)-periodic solution are established, its integral representation is given, and an estimate is obtained.

The problems of multiperiodic solutions of quasilinear systems with differentiation operators from subsections 1.4, 1.5 and 1.6 are investigated **in the second section** of the thesis. Operators of subsections 1.1, 1.2 and 1.3 are special cases of operators D_3 and D_3.

Sufficient conditions for the existence of a unique solution to the quasilinear systems $D_3 x = Ax + f(\zeta, x)$ in the non-critical case are given based on the method of compression mappings in the subsection 2.1. Further, the obtained results are generalized to the case of a system $D_3 x = P(\zeta)x + f(\zeta, x)$, when the corresponding homogeneous system holds the property of exponential dichotomy.

The quasilinear system $D_4 x = Ax + f(\tau, t, \zeta, x)$ is considered in the subsection 2.2. Here, the results of subsection 1.5 are extended to the considered quasilinear system. Conditions that guarantee the existence of a multiperiodic solution are obtained using the fixed point principle in the space of continuously differentiable functions.

Subsection 2.3 is devoted a presentation of one method for studying multiperiodic solutions of the one system of the form $D_5 x = Ax + \varepsilon^0 f(\tau, t) + \varepsilon\varphi(x)$, where the nonlinear function $\varphi(x)$ is a homogeneous form of lth degree; ε^0 and ε are positive parameters.

The main task of the subsection 2.3 is to study the question of the existence of multiperiodic solutions of this system by a method combining elements of the method KAM theory and of successive approximations. A new approach is proposed which the classical approximation method is applied. This method is based on the idea that a small parameter under perturbation is considered to varying according to an exponential-hyperbolic law. Sufficient conditions for the existence of the multiperiodic solutions of considered quasilinear systems with an exponential-hyperbolic varying small parameter are obtained on the basis of the proposed method, and its construction is realized by the method of successive approximations.