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Unpredictable oscillations of differential equations and neural networks

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CONTENTS

NORMATIVE REFERENCES

In this thesis the following normative references were used:

ГОСТ 7.1-2003. Bibliographic records. Bibliographic description. General requirements and rules of compilation.

ГОСТ 7.32-2001 (changes from 2006). Scientific research paper report. Structure and rules of preparation.

DESIGNATIONS AND ABBREVIATIONS

SICNNs	– Shunting inhibitory cellular neural networks
HNNs	– Hopfield type neural networks
CGNNs	– Cohen-Grossberg type neural networks
INNs	– inertial neural networks
BAM	– bidirectional associative memory

INTRODUCTION

The thesis researches. The thesis is devoted to the study of the unpredictable solutions of differential equations and unpredictable oscillations of neural networks.

Actuality of the thesis. The actuality of the thesis is due to the numerous applications of differential equations in solving problems of natural science and the widespread use of neural networks in the modern world. The thesis is based on the concepts of unpredictable functions, which were introduced by M. Akhmet and M.O. Fen [1-4]. We proved the existence of unpredictable solutions of differential equations, and unpredictable oscillations of neural networks.

The main results of the thesis have been published in peer reviewed journals, which confirms the actuality of the research.

Preliminaries. Oscillations are necessary attribute of various processes occurring in nature [5-12]. Of exceptional theoretical and practical importance are the oscillatory motions described by differential equations. In the literature a large number of results have been obtained for periodic, quasiperiodic, and almost periodic solutions of differential equations due to the established mathematical methods and important applications [13-17]. On the other hand, recurrent and Poisson stable solutions are also crucial for the theory of differential equations [18-24].

The founders of the theory of non-linear oscillations are H. Poincare [25, 26] and A.M. Lyapunov [27], who created a mathematical apparatus suitable for the study of nonlinear systems. The theory of non-linear dynamics focused mainly on periodic motions. The first functions which can be considered still as "periodic" and sufficiently determined for strict mathematical analysis were quasiperiodic functions introduced and investigated by P. Bohl [28, 29] and E. Esclangon [30] independently. The fundamental papers of H. Bohr [31-33] provided the theory of almost periodic functions, which we call as H. Bohr almost periodic functions nowadays. Then different approaches to almost periodicity were found by N.N. Bogolyubov [34], A.S. Besicovitch [35], S. Bochner [36], V.V. Stepanov [37], and others. The almost periodic functions are of great importance for development of harmonic analysis on groups, Fourier series and integrals on groups. The paper [38] published by S. Bochner provided extension of the theory of almost periodic functions with values in a Banach space. The first paper on the existence of almost periodic solutions was written by H. Bohr and O. Neugebauer [39], and nowadays the theory of almost periodic equations has been developed in connection with problems of differential equations, stability theory, and dynamical systems. The list of the applications of the theory has been essentially extended and includes not only ordinary differential equations and classical dynamical systems, but also wide classes of partial differential equations and equations in Banach spaces [40]. The concepts of recurrent motions and Poisson stable points are classical notions central to the qualitative theory of motions for dynamical systems. Poisson stable points was considered by H. Poincare as the main element in the description of complexity in celestial dynamics.

The foundation of the research of non-linear oscillations in Kazakhstan were laid by V.H. Kharasakhal [41] and O.A. Zhautykov [42, 43]. D.U. Umbetzhanov and his colleagues, intensively investigated almost periodic and multi-periodic solutions for differential and evolution systems [44-49]. Nowadays, Kazakhstan mathematicians continue to make significant contribution to the analysis of oscillations [50-53].

In recent decades, researchers have focused on studying oscillations in neural networks. Neural networks were created to study brain activity. There are many models of neural networks, which mathematically described by recurrent and differential equations. For example, Hopfield type neural networks, shunting inhibitory cellular neural networks, Cohen-Grossberg neural networks, inertial neural networks are investigated.

Oscillatory neural networks are effective for image recognition [54], and in activating network states associated with memory recall [55]. It is natural that neural oscillations became the core of interdisciplinary research that unites neuroscience, psychophysics, biophysics, cognitive psychology, and computational modeling [56-65].

This is why, many researchers are studying periodic, almost periodic, and exponential stability for neural networks considering the input-output mechanism [66-71].

Recently, study of chaotic oscillations in neural networks starts to be of significant interest [72-75]. Solutions of the chaotic systems are irregular, and this is reflected by data related to experiments and observations [76-81].

A few years ago, M. Akhmet and M.O. Fen introduced the concepts of unpredictable points and unpredictable functions and thereby significantly expanded the boundaries of the classical theory of dynamical systems, founded by H. Poincare and G. Birkhoff [82]. An unpredictable point is a modernization of the Poisson stable point. In paper, it was proved that the quasi-minimal set is chaotic set, if the Poisson stable point also admit the unpredictability property. Thus, the presence of chaos in a dynamic system is determined by the presence of only one point - unpredictable. Unpredictable functions were defined as unpredictable points in the Bebutov dynamical system with the only difference that the topology of convergence on compact sets of the real axis is used instead of the metric space. The use of such convergence makes it possible to significantly simplify the problem of proving the existence of unpredictable solutions for differential equations. And one can completely remain in the field of the theory of differential equations without mentioning the original or related results in the theory of dynamical systems or chaos.

The unpredictable points are used by A. Miller [83], R. Thakur and R. Das [84] in topological spaces where they considered Poincare chaos, strongly Ruelle-Takens chaos, and strongly Auslander-York chaos.

The goal of this study. The goal of the thesis is to use the method and theoretical basis laid down in the articles by M. Akhmet and M.O. Fen, to prove that linear and quasilinear differential equations admit unpredictable solutions. Moreover, research the existence of unpredictable oscillations for SICNNs and INNs.

The research problems. The main tasks of the study are as follows:

a) the existence and uniqueness of stable unpredictable solutions for linear and quasilinear ordinary differential equations;

b) the existence, uniqueness and asymptotic stability of unpredictable and strongly unpredictable oscillations in neural networks;

c) examples and numerical simulations confirming the validity of the theoretical results.

The objective of the research. The main objective is the unpredictable oscillations of differential equations and neural networks.

The subject of the study. The subject of the research is the existence and uniqueness of asymptotically stable unpredictable and strongly unpredictable solutions of differential equations and neural networks.

The scientific novelties. The novelties of the thesis are as follows:

a) the existence and uniqueness of asymptotically stable unpredictable and strongly unpredictable solutions of differential equations;

c) the existence of asymptotically stable unpredictable and strongly unpredictable oscillations of the neural networks;

d) examples and numerical simulations confirming the feasibility of the theoretical results.

The results of the thesis which are taken out on defense:

– the theorem on the existence and uniqueness of uniformly asymptotically stable unpredictable solutions of ordinary linear differential equations;

– the theorem on the existence and uniqueness of uniformly exponentially stable unpredictable solutions of ordinary quasilinear differential equations;

– the theorem on the existence of uniformly exponentially stable unpredictable solutions of SICNNs;

– the theorem on the existence and uniqueness of asymptotically stable strongly unpredictable unpredictable solutions of SICNNs;

– the theorem on the existence and uniqueness of asymptotically stable unpredictable solutions of INNs;

– the ways to construction unpredictable functions.

The reliability and validity. In the thesis, methods and results of the theory of functional analysis, algebra and differential equations are widely used.

The theoretical and practical significance of the results. The scientific significance of the research lies in the fact that the results will become the basis for the study of unpredictable oscillations for different types of differential equations, such as, impulsive differential equations, partial differential equations and others. The control of unpredictable oscillations will allow them to be used in medicine, biology, cryptography and many other fields.

The relation of thesis to other research. The thesis was carried out as part of the grant research project of the Ministry of Education and Science of the Republic of Kazakhstan on fundamental investigations in the field of natural sciences «Cellular neural networks with continuous/discrete time and singular perturbations» (No. AP 05132573, 2018-2020).

The personal contribution of the author. All the results of the thesis are obtained by the author. The participation of co-authors and scientific consultants consists in setting goals and discussing the results.

Approbation of the received results. The main results of the thesis were reported and discussed at the following events:

– VIII International Scientific Conference «Problems of Differential Equations, Analysis and Algebra» (Aktobe, Kazakhstan, November 1, 2018);

– V International Research Symposium on the Turkic World (Almaty, Kazakhstan, October 11-13, 2018);

– IV International Scientific and Practical Conference "Computer Science and Applied Mathematics" (Almaty, Kazakhstan, September 25-28, 2019);

– 11th International Conference on Information Management and Engineering (ICIME 2019) (London, UK, September 16-18, 2019).

Publications. On the topic of the dissertation, 11 articles were published, including 5 publication in a ranking scientific journal indexed in the Scopus database, 2 publications in scientific journals included in the list recommended by the Committee for Control in the Sphere of Education and Science (formerly CCSES) of the Ministry of Education and Science of the Republic of Kazakhstan for publication of the main scientific results of scientific activities, 4 publications in the materials of the international conferences, including 1 publication in the materials of a foreign conference indexed in the database Scopus.

Structure and thesis volume. The thesis consists of introduction, 2 chapters, conclusion and list of references from 112 publications. The numbering of formulas, theorems and definitions is three-digit: the first number means the number of the chapter, the second is the number of the section, the third is the own number of the formula, theorem, definition inside the section. The thesis is set out on 77 pages.

Summary of work.

The chapter 1 devoted to unpredictable solutions of differential equations. In the first section the basic definitions of unpredictable functions are presented.

Definition 0.1. [4, p. 658] A uniformly continuous and bounded function $\vartheta: \mathbb{R} \to \mathbb{R}^p$ is unpredictable if there exist positive numbers ε_0, δ and sequences $\{t_n\}, \{u_n\}$ both of which diverge to infinity such that $\|\vartheta(t+t_n) - \vartheta(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\vartheta(t+t_n) - \vartheta(t)\| \geq \varepsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$.

The convergence of the sequence $\vartheta(t+t_n)$ is said to be *Poisson stability of the unpredictable function* or simply *Poisson stability* as well as existence of the numbers ε_0, δ and the sequence u_n is allow *the unpredictable property of the unpredictable function*.

The Definition 0.1 implies that some coordinates may not be unpredictable scalar valued functions. That is why, it is of great importance for applications to consider motions which are unpredictable in all state dimensions, that is strongly unpredictable functions.

Definition 0.2. A uniformly continuous and bounded function $v: \mathbb{R} \to \mathbb{R}^p$,

$v = (v_1, v_2, \ldots, v_p)$, is strongly unpredictable if there exist positive numbers ε_0, δ and sequences $\{t_n\}, \{u_n\}$, both of which diverge to infinity such that $v(t + t_n) \to v(t)$ as $n \to \infty$ uniformly on compact sets of $\mathbb{R}$ and $|v_i(t + t_n) - v_i(t)| \geq \varepsilon_0$ for all $i = 1,2\ldots, p$, $t \in [u_n - \delta, u_n + \delta]$, and $n \in \mathbb{N}$.

The properties of unpredictable functions are given. The example of the unpredictable function is constructed.

In the second section proved the existence of uniformly asymptotically stable unpredictable solution of linear differential equations:

$$x'(t) = Ax(t) + g(t), \tag{0.1}$$

where $x \in \mathbb{R}^p$ and $g: \mathbb{R} \to \mathbb{R}^p$ is an unpredictable function. All eigenvalues of the constant matrix $A \in \mathbb{R}^{p \times p}$ have nonzero real parts.

The main object of the third section is the system of quasilinear differential equations:

$$x'(t) = Ax(t) + f(x(t)) + g(t), \tag{0.2}$$

where $x(t) \in \mathbb{R}^p$, p is a fixed natural number, the function $f: \mathbb{R}^p \to \mathbb{R}^p$ is continuous in all of its arguments, and all eigenvalues of the constant matrix $A \in \mathbb{R}^{p \times p}$ have nonzero real parts. Moreover, the function $g: \mathbb{R} \to \mathbb{R}^p$ is unpredictable.

It is known that one can find a regular $p \times p$ matrix B such that the transformation $x = By$ reduces the system (0.2) to the system:

$$y'(t) = Cy(t) + F(y) + G(t), \tag{0.3}$$

where $C = B^{-1}AB, F(y) = B^{-1}f(By)$, and $G(t) = B^{-1}g(t)$. In system (0.3), the matrix C is of the form diag(C_-, C_+), where the eigenvalues of the $q \times q$ matrix C_- and $(p-q) \times (p-q)$ matrix C_+ respectively have negative and positive real parts.

One can confirm that there exist numbers $K \geq 1$ and $\alpha > 0$ such that $||e^{C_- t}|| \leq Ke^{-\alpha t}$ for all $t \geq 0$ and $||e^{C_+ t}|| \leq Ke^{\alpha t}$ for all $t \leq 0$.

The following conditions are required.

(C1) There exists a positive number L_f such that $\left||f(x_1) - f(x_2)|\right| \leq$ $\leq L_f ||x_1 - x_2||$ for all $x_1, x_2 \in \mathbb{R}^p$.

(C2) $\frac{2}{\alpha} K(||B||||B^{-1}||L_f + 1) < 1$;

Theorem 0.1. *Suppose that conditions* $(C1), (C2)$ *are valid, then system (0.2) possesses a unique unpredictable solution. Moreover, the unpredictable solution is uniformly exponentially stable if all eigenvalues of the matrix A have negative real parts.*

In the fourth section we extend Definition 0.1 to the class of functions with several independent variables. The following new definitions will be of use.

Definition 0.3. A continuous and bounded function $f(t,x)\colon \mathbb{R}\times G \to \mathbb{R}^p$, $f = (f_1, f_2, \ldots, f_p)$, $G \subset \mathbb{R}^p$ is a bounded domain, is unpredictable in t if it is uniformly continuous in t and there exist positive numbers ε_0, δ and sequences $\{t_n\}, \{u_n\}$, both of which diverge to infinity such that $\sup_G ||f(t+t_n,x) - f(t,x)|| \to 0$ as $n \to \infty$ uniformly on compact sets in $\mathbb{R}$ and $||f(t+t_n,x) - f(t,x)|| \geq \varepsilon_0$ for $t \in [u_n - \delta, u_n + \delta]$, $x \in G$ and $n \in \mathbb{N}$.

Definition 0.4. A continuous and bounded function $f(t,x)\colon \mathbb{R}\times G \to \mathbb{R}^p$, $f = (f_1, f_2, \ldots, f_p)$, $G \subset \mathbb{R}^p$ is a bounded domain, is strongly unpredictable in t if it is uniformly continuous in t and there exist positive numbers ε_0, δ and sequences $\{t_n\}$, $\{u_n\}$ both of which diverge to infinity such that $\sup_G \left|\left|f(t+t_n,x) - f(t,x)\right|\right| \to 0$ as $n \to \infty$ uniformly on compact sets in $\mathbb{R}$ and $|f_i(t+t_n,x) - f_i(t,x)| \geq \varepsilon_0$ for all $i = 1,2\ldots,p$, $(t,x) \in [u_n - \delta, u_n + \delta] \times G$, and $n \in \mathbb{N}$.

The following system of differential equations is considered:

$$x'(t) = Ax(t) + f(t,x), \tag{0.4}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^p$, p is a fixed natural number, all eigenvalues of the constant matrix $A \in \mathbb{R}^{p\times p}$ have negative real parts, $f\colon \mathbb{R}\times G \to \mathbb{R}^p, f = (f_1, f_2, \ldots, f_p)$, $G = \{x \in \mathbb{R}^p, ||x|| < H\}$, where H is a positive number. It is true that there exist two real numbers $K \geq 1$ and $\gamma < 0$ such that $||e^{At}|| \leq Ke^{\gamma t}$ for all $t \geq 0$.

One can see that the main difference between system (0.2) and system (0.4) is that the perturbation in the former one is less general than that of the latter one.

The following conditions will be needed:

$(C1)$ the function $f(t,x)$ is strongly unpredictable in the sense of Definition 0.4.

The Definition 0.4 implies that there exists a positive number M such that $\sup_{\mathbb{R}\times G} ||f(t,x)|| = M < \infty$;

$(C2)$ there exists a positive constant L such that the function $f(t,x)$ satisfies the inequality $||f(t,x_1) - f(t,x_2)|| \leq L||x_1 - x_2||$ for all $t \in \mathbb{R}, x_1, x_2 \in G$;

$(C3)$ $\gamma < -\frac{KM}{H}$;

$(C4)$ $\gamma < -KL$.

Theorem 0.4. *If conditions $(C1)-(C4)$ are fulfilled, then the system (0.4) admits a unique uniformly exponentially stable strongly unpredictable solution.*

Moreover, it is proved that if the function $f(t,x)$ is unpredictable in the sense of Definition 0.3, then the system (0.4) admits a unique uniformly exponentially stable unpredictable solution.

The theoretical results confirmed by examples and graphical illustrations.

In the second chapter we discuss about unpredictable oscillations in neural networks.

In the first section SICNNs are considered, which have been introduced by A. Bouzerdoum and R. Pinter. In its original formulation [85], the SICNNs model is a two-dimensional grid of processing cells. Let C_{ij} denote the cell at the *(i, j)* position of the lattice. Denote by $N_r(ij)$ the r – neighborhood C_{ij}, such that:

$$N_r(ij) = \{C_{kp}\colon \max(|k-i|, |p-j|) \le r, 1 \le k \le m, 1 \le p \le n\},$$

where m and n are fixed natural numbers. In SICNNs, neighboring cells exert mutual inhibitory interactions of the shunting type. The dynamics of the cell C_{ij} is described by the following nonlinear ordinary differential equation:

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp} f(x_{kp}(t))x_{ij}(t) + v_{ij}(t), \qquad (0.5)$$

where x_{ij} is activity of the cell C_{ij}, the constant a_{ij} represents the passive decay rate of the cell activity;

$C_{ij}^{kp} \ge 0$ is the connection or coupling strength of postsynaptic activity of the cell C_{kp} transmitted to the cell C_{ij} and the activation $f(x_{kp})$ is a positive continuous function representing the output or firing rate of the cell C_{kp}, $v_{ij}(t)$ is the external input to the cell C_{ij}.

Let us denote by $\mathcal{A}$ the set of functions $u(t) = (u_{11}, \dots, u_{1n}, \dots, u_{m1} \dots, u_{mn})$, $t, u_{ij} \in \mathbb{R}, i = 1,2,\dots,m, j = 1,2,\dots,n,$ where $m, n \in \mathbb{N}$, such that:

($\mathcal{A}1$) functions $u(t)$ are uniformly continuous and there exists a positive number H such that $||u||_1 < H$ for all $u(t) \in \mathcal{A}$;

($\mathcal{A}2$) there exists a sequence $t_p, t_p \to \infty$ as $p \to \infty$ such that for each $u(t) \in \mathcal{A}$ the sequence $u(t + t_p)$ uniformly converges to $u(t)$ on each closed and bounded interval of the real axis.

The following assumptions will be needed

($C1$) the function $v(t) = (v_{11}, \dots, v_{1n}, \dots, v_{m1} \dots, v_{mn}), t, v_{ij} \in \mathbb{R}, i = 1, \dots, m,$ $j = 1,\dots,n,$ in system (0.5) belongs to $\mathcal{A}$ and is unpredictable such that there exist positive numbers $\delta, \varepsilon_0 > 0$ and a sequence $t_p \to \infty$ as $p \to \infty$, which satisfy $||v(t + t_p) - v(t)|| \ge \varepsilon_0$ for all $t \in [s_p - \delta, s_p + \delta]$, and $p \in \mathbb{N}$.

($C2$) for the rates we assume that $\gamma = \min_{(i,J)} a_{ij} > 0$ and $\bar{\gamma} = \max_{(i,j)} a_{ij}$;

($C3$) there exist positive numbers m_{ij} and m_f such that $\sup_{t\in\mathbb{R}} |v_{ij}| \le m_{ij}$ for all $i = 1, \dots, m, j = 1, \dots, n,$ and $\sup_{|s|<H} |f(s)| \le m_f$;

($C4$) there exists Lipschitz constant L such that $|f(s_1) - f(s_2)| \le L|s_1 - s_2|$ for all $s_1, s_2, |s_1| < H, |\, s_2| < H$;

$(C5)\left(LH+m_f\right)\max_{(i,j)}\sum_{C_{kp}\in N_r(i,j)}C_{ij}^{kp}<\gamma$ for all $i=1,\ldots,m, j=1,\ldots,n.$

The main result of the present section is mentioned in the next theorem.

Theorem 0.5. *Suppose that conditions* $(C1)-(C5)$ *are valid, then the system (0.5) possesses a unique asymptotically stable unpredictable solution.*

In the second section we consider following SICNNs

$$\frac{dx_{ij}}{dt}=-b_{ij}x_{ij}-\sum_{D_{kp}\in N_r(i,j)}D_{ij}^{kp}f(x_{kp}(t))x_{ij}(t)+g_{ij}(t), \tag{0.6}$$

with strongly unpredictable perturbations.

We denote by $\mathcal{B}$ the set of functions $u(t)=(u_{11},\ldots,u_{1n},\ldots,u_{m1}\ldots,u_{mn})$, $t,u_{ij}\in\mathbb{R}, i=1,2,\ldots,m, j=1,2,\ldots,n,$ where $m,n\in\mathbb{N}$, such that:

$(\mathcal{B}1)$ functions $u(t)$ are uniformly continuous;

$(\mathcal{B}2)$ there exists a positive number H such that $||u||_1<H$ for all $u(t)\in\mathcal{B}$;

$(\mathcal{B}3)$ there exists a sequence $t_p, t_p\to\infty$ as $p\to\infty$ such that for each $u(t)\in\mathcal{B}$ the sequence $u(t+t_p)$ uniformly converges to $u(t)$ on each closed and bounded interval of the real.

The following conditions are needed

$(D1)$ the function $g(t)=(g_{11},\ldots,g_{1n},\ldots,g_{m1}\ldots,g_{mn}), t, g_{ij}\in\mathbb{R}, i=1,2,\ldots,m,$

$j=1,2,\ldots,n,$ in system (0.6) belongs to $\mathcal{B}$ and is strongly unpredictable such that there exist positive numbers $\delta,\varepsilon_0>0$ and a sequence $t_p\to\infty$ as $p\to\infty$, which satisfy $|g_{ij}(t+t_p)-g_{ij}(t)|\geq\varepsilon_0$ for all $t\in[s_p-\delta,s_p+\delta], i=1,\ldots,m, j=1,\ldots,n,$ and $p\in\mathbb{N}$.

$(D2)$ $\gamma\leq b_{ij}\leq\bar{\gamma}$, where $\gamma,\bar{\gamma}$ are positive numbers;

$(D3)$ $|g_{ij}(t)|\leq m_{ij}$, where m_{ij} are positive numbers, for all $i=1,\ldots,m, j=1,\ldots,n,$ and $t\in\mathbb{R}$;

$(D4)$ $|f(s)|\leq m_f$, for $|s|<H$ and some constant $m_f>0$;

$(D5)$ there exists Lipschitz constant L such that $|f(s_1)-f(s_2)|\leq L|s_1-s_2|$ for all $s_1,s_2,|s_1|<H,|s_2|<H$;

$(D6)$ $m_f\sum_{D_{kp}\in N_r(i,j)}D_{ij}^{kp}<b_{ij}$, for each $i=1,\ldots,m, j=1,\ldots,n;$

$(D7)$ $\dfrac{m_{ij}}{b_{ij}-m_f\sum_{D_{kp}\in N_r(i,j)}D_{ij}^{kp}}<H$, for all $i=1,\ldots,m, j=1,\ldots,n;$

$(D8)$ $(LH+m_f)\max_{(i,j)}\sum_{D_{kp}\in N_r(i,j)}D_{ij}^{kp}<\gamma.$

Theorem 0.6. *Suppose that conditions* $(D1)-(D8)$ *are valid, then the system (0.6) possesses a unique asymptotically stable strongly unpredictable solution.*

In the third section the following INN is considered:

$$\frac{d^2x_i(t)}{dt^2} = -a_i\frac{dx_i(t)}{dt} - b_ix_i(t) + \sum_{j=1}^{p} c_{ij}f_j(x_j(t)) + v_i(t), \tag{0.7}$$

where $t, x_i \in \mathbb{R}, i = 1,2, \ldots, p, p$ denotes the number of neurons in the network;

$x_i(t)$ with $i = 1,2, \ldots, p$, corresponds to the state of the unit i at time t; the second derivative is called an inertial term;

$b_i > 0, a_i > 0$ are constants;

f_i with $i = 1, \ldots, p$, denote the measures of activation to its incoming potentials of ith neuron; c_{ij} for all $i, j = 1,2, \ldots, p$, are constants, which denote the synaptic connection weight of the unit j on the unit i;

$v_i(t)$ are external inputs on the ith neuron at time t.

We assume that the coefficients c_{ij} are real, the activation functions $f_i: \mathbb{R} \to \mathbb{R}$ are continuous functions satisfy the following condition:

(I1) $|f_i(x_1) - f_i(x_2)| \le L_i|x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}$, where $L_i > 0$ are Lipschitz constant, for all $i = 1,2, \ldots, p$, and $\max_{1\le i\le p} L_i = L$.

By introducing the following variable transformation

$$y_i(t) = \xi_i\frac{dx_i(t)}{dt} + \zeta_ix_i(t), i = 1, \ldots, p, \tag{0.8}$$

the neural network (0.7) can be written as

$$\frac{dx_i(t)}{dt} = -\frac{\zeta_i}{\xi_i}x_i(t) + \frac{1}{\xi_i}y_i(t), \tag{0.9}$$

$$\frac{dy_i(t)}{dt} = -\left(a_i - \frac{\zeta_i}{\xi_i}\right)y_i(t) - \left(\xi_ib_i - \zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right)\right)x_i(t) +$$

$$+\xi_i\sum_{j=1}^{p} c_{ij}f_j(x_j(t)) + \xi_iv_i(t), \tag{0.10}$$

The following conditions will be needed:

(I2) the functions $v_i(t), i = 1, \ldots, p$, in system (0.7) are unpredictable;

(I3) there exists a positive number H and M_f such that $|f_i(s)| \le M_f, i = 1, \ldots, p,\ |s| < H$.

Moreover, we assume that for positive real numbers ζ_i and ξ_i, $i = 1, \ldots, p$ the following inequalities are valid:

(I4) $a_i > \frac{\zeta_i}{\xi_i} + \xi_i, \zeta_i > \xi_i > 1;$

(I5) $(a_i - \frac{\zeta_i}{\xi_i}) - (|\zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + \xi_i) > 0;$

(I6) $\frac{\xi_i M_f \sum_{j=1}^{p} c_{ij}}{(a_i - \frac{\zeta_i}{\xi_i}) - (|\zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + \xi_i)} < H;$

(I7) $\frac{1}{(a_i - \frac{\zeta_i}{\xi_i})}(|\zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + L\xi_i \sum_{j=1}^{p} c_{ij}) < 1;$

(I8) $\max_i(\frac{1}{\xi_i}, |\zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + L\xi_i \sum_{j=1}^{p} c_{ij}) < \min_i(\frac{\zeta_i}{\xi_i}, a_i - \frac{\zeta_i}{\xi_i}).$

The main theorem is proved.

Theorem 0.7. *Assume that conditions* $(I1) - (I8)$ *are fulfilled. Then the system (0.7) admits a unique asymptotically stable strongly unpredictable solution.*

Illustrative examples concerning unpredictable oscillations of neural networks are provided.

1 UNPREDICTABLE SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS

1.1 Unpredictable functions

Throughout the thesis, $\mathbb{R}, \mathbb{N}$ and $\mathbb{Z}$ will stand for the set of real, natural and integer numbers, respectively. Additionally, and the norm $||v||_1 = \sup_{t\in\mathbb{R}} ||v(t)||$, where $||v|| = \max_{1\le j\le p} |v_i|, v = (v_1, \ldots, v_p), t, v_i \in \mathbb{R}$ $i = 1,2, \ldots, p,$ $p \in \mathbb{N},$ will be used.

The following definition is one of the main in our study.

Definition 1.1.1. [4, p. 658] A uniformly continuous and bounded function $\vartheta: \mathbb{R} \to \mathbb{R}^m$ is unpredictable if there exist positive numbers ε_0, δ and sequences $\{t_n\}, \{u_n\}$ both of which diverge to infinity such that $\|\vartheta(t + t_n) - \vartheta(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\vartheta(t + t_n) - \vartheta(t)\| \ge \varepsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$.

The convergence of the sequence $\vartheta(t + t_n)$ is said to be *Poisson stability of the unpredictable function* or simply *Poisson stability* as well as existence of the numbers ε_0, δ and the sequence u_n is allow *the unpredictable property of the unpredictable function*.

The Definition 1.1.1 implies that some coordinates may not be unpredictable scalar valued functions. That is why, it is of great importance for applications to consider motions which are unpredictable in all state dimensions, that is strongly unpredictable functions.

Definition 1.1.2. A uniformly continuous and bounded function $v: \mathbb{R} \to \mathbb{R}^p$, $v = (v_1, v_2, \ldots, v_p)$, is strongly unpredictable if there exist positive numbers ε_0, δ and sequences $\{t_n\}, \{u_n\}$, both of which diverge to infinity such that $v(t + t_n) \to v(t)$ as $n \to \infty$ uniformly on compact sets of $\mathbb{R}$ and $|v_i(t + t_n) - v_i(t)| \ge \varepsilon_0$ for all $i = 1,2\ldots, p$, $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$.

It can easily show that any strongly unpredictable function is an unpredictable one as in Definition 1.1.1, but not vice versa. Moreover, each of the functions $v_i(t)$, $i = 1,2\ldots, p$, in the Definition 1.1.2 are unpredictable in the sense of the Definition 1.1.1.

The properties of unpredictable function

1. *The multiplication of a constant with an unpredictable function is an unpredictable function.*

2. *If the function $\phi(t): \mathbb{R} \to \mathbb{R}$ is unpredictable, then the function $\phi(t) + C$, where C is a constant, is also unpredictable.*

Proof. There exist positive numbers ε_0, δ and sequences $\{t_n\}, \{u_n\}$ both of which diverge to infinity such that $\|\phi(t + t_n) - \phi(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\phi(t + t_n) - \phi(t)\| \ge \varepsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$. Let us denote $\omega(t) = \phi(t) + C$. Then we have that $\|\omega(t + t_n) - \omega(t)\| = \|\phi(t + t_n) - \phi(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\omega(t + t_n) - \omega(t)\| = \|\phi(t + t_n) - \phi(t)\| \ge \varepsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$. Therefore, the function $\phi(t) + C$ is unpredictable.

3. *Suppose that* $\phi(t): \mathbb{R} \to \mathbb{R}$ *is an unpredictable function. Then the function* $\phi^3(t)$ *is unpredictable.*

Proof. One can find numbers $\varepsilon_0 > 0, \delta > 0$ and sequences $\{t_n\}, \{u_n\}$ both of which diverge to infinity such that $\|\phi(t+t_n) - \phi(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|\phi(t+t_n) - \phi(t)\| \geq \varepsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$. It is easy to check that $\|\phi^3(t+t_n) - \phi^3(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$, since it follows from the uniform continuity of the cubic function on a compact set.

Fix a natural number n. Let us show that for $t \in [u_n - \delta, u_n + \delta]$ the inequality $\|\phi(t+t_n) - \phi(t)\| \geq \varepsilon_0$ implies $\|\phi^3(t+t_n) - \phi^3(t)\| \geq \varepsilon_0^3/4$.

We have that:

$$|\phi^3(t+t_n) - \phi^3(t)| = \frac{1}{2}|\phi(t+t_n) - \phi(t)|[\phi^2(t+t_n) + \phi^2(t) +$$

$$+(\phi(t+t_n) + \phi(t))^2] \geq \tfrac{1}{2}(\phi^2(t+t_n) + \phi^2(t))\varepsilon_0.$$

Consider the function $F(a,b) = a^2 + b^2$ for $|a - b| \geq \varepsilon_0$. The minimum of F occurs at the points (a,b) with $|a| = |b| = \varepsilon_0/2$. Therefore, $|\phi^3(t+t_n) - \phi^3(t)| \geq$ $\geq \varepsilon_0^3/4$ for $t \in [u_n - \delta, u_n + \delta]$.

Next, we extend Definition 1.1.1 to the class of functions with several independent variables. The following new definition will be of use.

Definition 1.1.3. A continuous and bounded function $f(t,x): \mathbb{R} \times G \to \mathbb{R}^p$, $f = (f_1, f_2, \ldots, f_p)$, $G \subset \mathbb{R}^p$ is a bounded domain, is unpredictable in t if it is uniformly continuous in t and there exist positive numbers ε_0, δ and sequences $\{t_n\}, \{u_n\}$, both of which diverge to infinity such that $\sup_G ||f(t+t_n, x) - f(t,x)|| \to$ $\to 0$ as $n \to \infty$ uniformly on compact sets in $\mathbb{R}$ and $||f(t+t_n, x) - f(t,x)|| \geq \varepsilon_0$ for $t \in [u_n - \delta, u_n + \delta], x \in G$ and $n \in \mathbb{N}$.

We consider nonlinear perturbations, which are functions unpredictable in the time variable. Thus, in the present section we have significantly enlarged the set of systems, which can be investigated for unpredictable solutions. To this end, we shall need the following new notion.

Definition 1.1.4. A continuous and bounded function $f(t,x): \mathbb{R} \times G \to \mathbb{R}^p$, $f = \left(f_1, f_2, \ldots, f_p\right)$, $G \subset \mathbb{R}^p$ is a bounded domain, is strongly unpredictable in t if it is uniformly continuous in t and there exist positive numbers ε_0, δ and sequences $\{t_n\}$, $\{u_n\}$, both of which diverge to infinity such that $\sup_G \left|\left|f(t+t_n, x) - f(t,x)\right|\right| \to 0$ as $n \to \infty$ uniformly on compact sets in $\mathbb{R}$ and $|f_i(t+t_n, x) - f_i(t,x)| \geq \varepsilon_0$ for all $i = 1,2\ldots,p$, $(t,x) \in [u_n - \delta, u_n + \delta] \times G$, and $n \in \mathbb{N}$.

An example of unpredictable function

Let us take into account the logistic map

$$\lambda_{i+1} = F_\mu(\lambda_i), \tag{1.1.1}$$

where $i \in \mathbb{Z}$ and $F_\mu(s) = \mu s(1-s)$. The interval [0,1) is invariant under the iterations of (1.1.1) for $\mu \in (0,4]$ [86]. It was shown in [3, p. 88] that the logistic map (1.1.1) possesses an unpredictable solution for each $\mu \in [3 + (2/3)^{1/2}, 4]$.

Let $\psi_i, i \in \mathbb{Z}$, be an unpredictable solution of the logistic map (1.1.1) with $\mu =$ $= 3.92$ inside the unit interval (0,1]. There exist a positive number ρ and sequences ζ_p, η_p, both of which diverge to infinity such that $|\psi_{i+\eta_p} - \psi_{\eta_p}| \to 0$ as $p \to \infty$ for each i in bounded intervals of integers and $|\psi_{\zeta_p+\eta_p} - \psi_{\eta_p}| \geq \rho$ for each $p \in \mathbb{N}$.

Define the following integral:

$$\Theta(t) = \int_{-\infty}^{t} e^{-3(t-s)} \Omega(s) ds, \tag{1.1.2}$$

where $\Omega(t)$ is a piecewise constant function defined on the real axis through the equation $\Omega(t) = \psi_i$ for $t \in [i, i+1], i \in \mathbb{Z}$. In figure 1 the graph of function $\Omega(t)$ is shown.

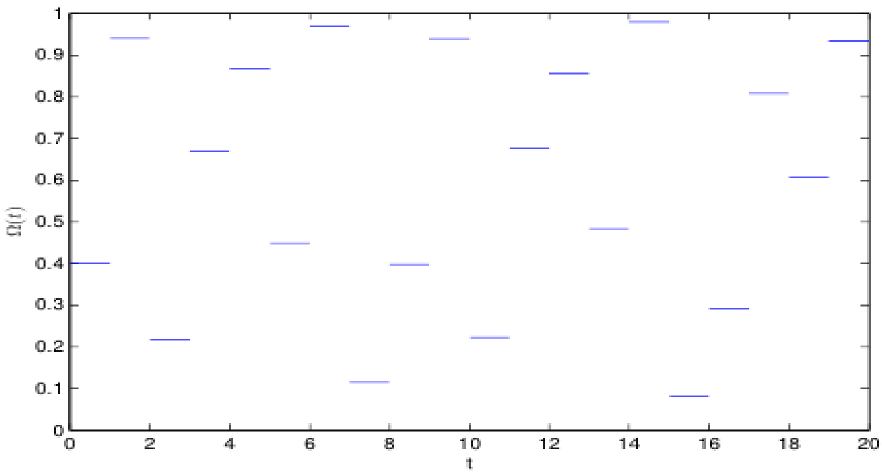

Figure 1 – The graph of piecewise constant function $\Omega(t)$

It is worth noting that $\Theta(t)$ is bounded on the whole real axis such that $\sup_{t \in \mathbb{R}} |\Theta(t)| \leq 1/3$.

Next, we will show that function $\Theta(t)$ is an unpredictable function. Consider a fixed bounded and closed interval $[\alpha, \beta]$, of the axis and a positive number ε. Without loss of generality, one can assume that α and β are integers. Now, we fix a positive number ξ and and integer $\gamma < \alpha$, which satisfy the following inequalities $\frac{2}{3} e^{-3(\alpha-\gamma)} < \frac{\varepsilon}{2}$ and $\frac{\xi}{3}[1 - e^{-3(\beta-\gamma)}] < \frac{\varepsilon}{2}$. Let p be a large enough number such that $|\Omega(t + \zeta_p) - \Omega(t)| < \xi$ on $[\gamma, \beta]$. Then for all $t \in [\alpha, \beta]$ we obtain that:

$$|\Theta(t + \zeta_p) - \Theta(t)| = |\int_{-\infty}^{t} e^{-3(t-s)} (\Omega(s + \zeta_p) - \Omega(s)) ds| =$$

$$= |\int_{-\infty}^{\gamma} e^{-3(t-s)}\left(\Omega(s+\zeta_p)-\Omega(s)\right)ds + \int_{\gamma}^{t} e^{-3(t-s)}(\Omega(s+\zeta_p)-\Omega(s))ds| \le$$

$$\le \int_{-\infty}^{\gamma} e^{-3(t-s)}2ds + \int_{\gamma}^{t} e^{-3(t-s)}\xi ds \le \frac{2}{3}e^{-3(\alpha-\gamma)} + \frac{\xi}{3}[1-e^{-3(\beta-\gamma)}] < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

Thus $|\Theta(t+\zeta_p)-\Theta(t)| \to 0$ as $p\to\infty$ uniformly on the interval $[\gamma,\beta]$.

It is true that $|\Omega(t+\zeta_p)-\Omega(t)| = |\psi_{\zeta_p+\eta_p}-\psi_{\eta_p}| \ge \rho, t\in[\eta_p,\eta_p+1]$ for all $p\in\mathbb{Z}$, and there exists a positive number $\kappa<1$ such that $\frac{2}{3}[1-e^{3\kappa}] \le \frac{\rho}{6}$.

Let us fix the number κ and $p\in\mathbb{N}$, and consider two alternative cases: (i) $|\Theta(\eta_p+\zeta_p)-\Theta(\eta_p)| < \frac{\rho}{4}$ and (ii) $|\Theta(\eta_p+\zeta_p)-\Theta(\eta_p)| \ge \frac{\rho}{4}$.

It is easily seen that the following relation holds:

$$\Theta(t+\zeta_p)-\Theta(t) =$$
$$= \Theta(\eta_p+\zeta_p)-\Theta(\eta_p) + \int_{\eta_p}^{t} e^{-3(t-s)}(\Omega(s+\zeta_p)-\Omega(s))ds. \quad (1.1.3)$$

(i) From the last relation we obtain that

$$\left|\Theta(t+\zeta_p)-\Theta(\mathrm{t})\right| \ge$$
$$\ge |\int_{\eta_p}^{t} e^{-3(t-s)}\left(\Omega(s+\zeta_p)-\Omega(s)\right)ds| - \left|\Theta(\eta_p+\zeta_p)-\Theta(\eta_p)\right| \ge$$

$$\ge \int_{\eta_p}^{t} e^{-3(t-s)}\rho ds - \frac{\rho}{4} \ge \frac{\rho}{3}-\frac{\rho}{4} = \frac{\rho}{12} \quad (1.1.4)$$

for $t\in[\eta_p,\eta_p+1)$.

(ii) Using the relation (1.1.3) we get that

$$\left|\Theta(t+\zeta_p)-\Theta(\mathrm{t})\right| \ge$$

$$\ge |\Theta(\eta_p+\zeta_p)-\Theta(\eta_p)| - |\int_{\eta_p}^{t} e^{-3(t-s)}(\Omega(s+\zeta_p)-\Omega(s))ds| \ge$$

$$\ge \frac{\rho}{4} - \int_{\eta_p}^{t} 2e^{-3(t-s)}ds \ge \frac{\rho}{4}-\frac{2}{3}(1-e^{-3\kappa}) \ge \frac{\rho}{12} \quad (1.1.5)$$

for $t\in[\eta_p,\eta_p+\kappa)$.

Thus, (1.1.4) and (1.1.5) prove finally that the function $\Theta(t)$ is unpredictable with positive $\varepsilon_0 = \frac{\rho}{12}, \delta = \frac{\kappa}{3}$ and sequences $t_p = \zeta_p, s_p = \eta_p + \frac{\kappa}{2}$.

Since we do not reliably know the initial value of the function $\Theta(t)$, we are not able to visualize it. For this reason, let's represent the function $\Theta(t)$, as follows:

$$\Theta(t) = \int_{-\infty}^{t} e^{-3(t-s)}\Omega(s)ds = e^{-3t}\Theta_0 + \int_0^t e^{-3(t-s)}\Omega(s)ds, \qquad (1.1.6)$$

where

$$\Theta_0 = \int_{-\infty}^{0} e^{3s}\Omega(s)ds.$$

Next, we will simulate the function

$$\Phi(t) = e^{-3t}\Phi_0 + \int_0^t e^{-3(t-s)}\Omega(s)ds, t \geq 0, \qquad (1.1.7)$$

where Φ_0 is a fixed number, which not necessarily equal to Θ_0. If we subtract the equality (1.1.7) from equality (1.1.6), we obtain $\Theta(t) - \Phi(t) = e^{-3t}(\Theta_0 - \Phi_0)$, $t \geq 0$. The last equation demonstrates that the difference $\Theta(t) - \Phi(t)$ exponentially diminishes. Consequently, the graph of the function $\Phi(t)$ approximates the graph of the function $\Theta(t)$, as time increases. Figure 2 shows the graph of the function $\Phi(t)$, defined by the initial value $\Phi(0) = 0.8$ accurate to 10^{-4}. Moreover, since $\sup_{t\in\mathbb{R}} |\Theta(t)| \leq 1/3$, then for $t \geq 2ln10 - \frac{1}{3}ln\frac{3}{2} \approx 3.71$ the function $\Phi(t)$ will coincide with $\Theta(t)$ to within 10^{-6}.

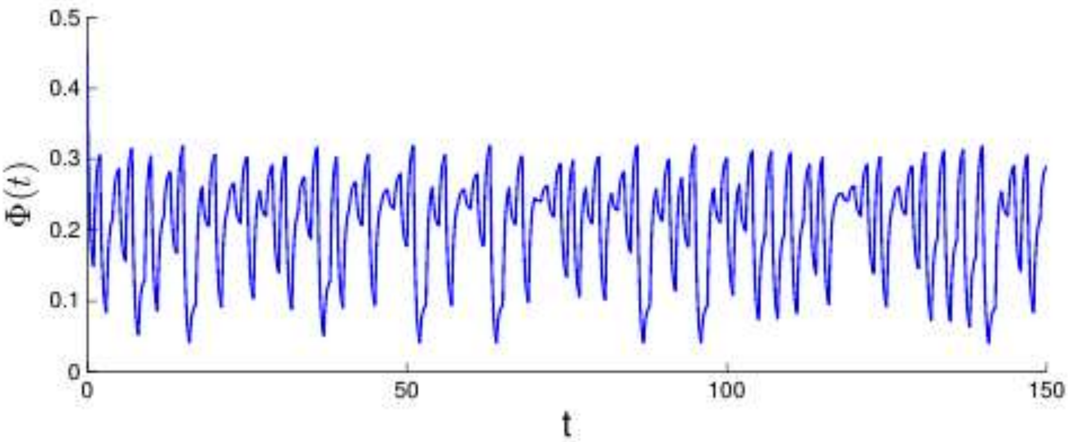

Figure 2 – The graph of function Φ(t), which approaches the unpredictable function Θ(t)

1.2 Hyperbolic linear systems

Let us consider the system of linear differential equations:

$$x'(t) = Ax(t) + g(t), \tag{1.2.1}$$

where $x \in \mathbb{R}^p$, and the function $g: \mathbb{R} \to \mathbb{R}^p$ is uniformly continuous and bounded. Moreover, all eigenvalues of the constant matrix $A \in \mathbb{R}^{p\times p}$ have nonzero real parts.

We will make use of the usual Euclidian norm, and the norm induced by the Euclidean norm for square matrices.

Assume that the following condition is valid.

(C) $Re\,\lambda_i < 0, i = 1,2,\dots,r$, and $Re\,\lambda_i > 0, i = r+1, r+2,\dots,p, 1 \le r < p$, where λ_i, $i = 1,2,\dots,p$, are the eigenvalues of the matrix A and $Re\,\lambda_i$ denotes the real part of λ_i.

One can find a nonsingular matrix B such that the substitution $x = By$ transforms system (1.2.1) to the equation:

$$y'(t) = B^{-1}ABy(t) + B^{-1}g(t), \tag{1.2.2}$$

with the block diagonal matrix of coefficients [87]. Therefore, we assume without loss of generality that the matrix A in system (1.2.1) is block diagonal such that $A = diag(A_-, A_+)$, where the eigenvalues of the matrices A_- and A_+ possess negative and positive real parts, respectively. There exist numbers $K \ge 1$ and $\alpha > 0$ such that $||e^{A_- t}|| \le Ke^{-\alpha t}$ for $t \ge 0$ and $||e^{A_+ t}|| \ge Ke^{\alpha t}$ for $t < 0$.

From equation (1.2.1), it implies that the following auxiliary assertion is needed.

Lemma 1.2.1. If the function $g(t)$ is unpredictable, then the function $f(t) = = B^{-1}g(t)$ is also unpredictable.

Proof. There exist positive numbers ε_0, δ and sequences $\{t_n\}, \{u_n\}$ both of which diverge to infinity such that $\|g(t+t_n) - g(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$ and $\|g(t+t_n) - g(t)\| \ge \varepsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$. Then we have that $\|f(t+t_n) - f(t)\| = \left|\left|B^{-1}\big(g(t+t_n) - g(t)\big)\right|\right| \le \le \left|\left|B^{-1}\right|\right| \cdot \left|\left|g(t+t_n) - g(t)\right|\right|$, this is why and $\|f(t+t_n) - f(t)\| \to 0$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{R}$. On the other hand, $\|f(t+t_n) - f(t)\| = = |B^{-1}(g(t+t_n) - g(t))|| \ge ||B^{-1}||\varepsilon_0$ for each $t \in [u_n - \delta, u_n + \delta]$ and $n \in \mathbb{N}$. The lemma is proved.

In what follows we will denote $g(t) = (g_-(t), g_+(t))$, where the vector-functions $g_-(t)$ and $g_+(t)$ are of dimensions *r* and *p-r*, respectively.

As it is known from the theory of differential equations [87, p. 150], system (1.2.1) admits a unique bounded on $\mathbb{R}$ solution $\varphi(t) = (\varphi_-(t), \varphi_+(t))$,

$$\varphi_-(t) = \int_{-\infty}^{t} e^{A_-(t-s)} g_-(s)ds, \qquad \varphi_+(t) = -\int_{t}^{\infty} e^{A_+(t-s)} g_+(s)ds, \tag{1.2.3}$$

if the function $g(t)$ is bounded on $\mathbb{R}$. One can confirm that $\sup_{t\in\mathbb{R}} ||\varphi(t)|| \leq \frac{2M_gK}{\alpha}$, where $M_g = \sup_{t\in\mathbb{R}} ||g(t)||$. Moreover, $\varphi(t)$ is periodic, quasi-periodic, or almost periodic if the perturbation function $g(t)$ is respectively of the same type.

The following theorem is concerned with the unpredictable solution of system (1.2.1).

Theorem 1.2.1. Assume that g(t) is an unpredictable function and condition (C) is valid. Then the system (1.2.1) possesses a unique unpredictable solution. Additionally, if all eigenvalues of the matrix A have negative real parts, then the unpredictable solution is uniformly asymptotically stable.

Proof. The boundedness of $g(t)$ implies that the system (1.2.1) admits a unique bounded solution $\varphi(t) = (\varphi_-(t), \varphi_+(t))$, which satisfies (1.2.3). Moreover, the bounded solution is uniformly asymptotically stable provided that all eigenvalues of the matrix *A* have negative real parts. Hence, it is sufficient to prove that $\varphi(t)$ is an unpredictable function.

The function $\varphi(t)$ is uniformly continuous since its derivative is bounded. According to the Poisson stability of $g(t)$, there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $||g(t+t_n) - g(t)|| \to 0$ uniformly on compact subsets of $\mathbb{R}$. One can easily find that:

$$\left||\varphi_-(t+t_n) - \varphi_-(t)\right|| = \left\|\int_{-\infty}^{t} e^{A_-(t-s)}[g_-(s+t_n) - g_-(s)]ds\right\| \leq$$

$$\leq \int_{-\infty}^{t} Ke^{\alpha_-(t-s)}\|g_-(s+t_n) - g_-(s)\|ds$$

and

$$\left||\varphi_+(t+t_n) - \varphi_+(t)\right|| = \left\|\int_{t}^{\infty} e^{A_+(t-s)}[g_+(s+t_n) - g_+(s)]ds\right\| \leq$$

$$\leq \int_{t}^{\infty} Ke^{\alpha(t-s)}\|g_+(s+t_n) - g_+(s)\|ds.$$

Fix an arbitrary positive number ε and a closed interval $[a,b], -\infty < a < b <$ $< \infty$, of the real axis. We will show that for sufficiently large n it is true that $\left||\varphi(t+t_n) - \varphi(t)\right|| < \varepsilon$ on $[a,b]$. Let us choose numbers $c < a, d > b, \xi > 0$ such that $\frac{2M_gK}{\alpha} e^{-\alpha(a-c)} \leq \frac{\varepsilon}{4}$, $\frac{2M_gK}{\alpha} e^{-\alpha(d-b)} \leq \frac{\varepsilon}{4}$, and $\frac{K\xi}{\alpha} \leq \frac{\varepsilon}{4}$.

Consider n sufficiently large such that $||g(t+t_n) - g(t)|| < \xi$ for $t \in [c,d]$. Then we have for all $t \in [a,b]$ that:

$$\left|\left|\varphi_-(t+t_n)-\varphi_-(t)\right|\right| \leq \int_{-\infty}^{c} Ke^{-\alpha(t-s)}\|g_-(s+t_n)-g_-(s)\|ds +$$

$$+\int_{c}^{t} Ke^{-\alpha(t-s)}\|g_-(s+t_n)-g_-(s)\|ds \leq$$

$$\leq \int_{-\infty}^{c} 2M_gKe^{-\alpha(t-s)}ds + \int_{c}^{t} K\xi e^{-\alpha(t-s)}ds < \frac{2M_gK}{\alpha}e^{-\alpha(a-c)} + \frac{K\xi}{\alpha} \leq \frac{\varepsilon}{2},$$

and similarly, one can show that:

$$\left|\left|\varphi_+(t+t_n)-\varphi_+(t)\right|\right| \leq \int_{t}^{d} Ke^{\alpha(t-s)}\|g_+(s+t_n)-g_+(s)\|ds +$$
$$+\int_{d}^{\infty} Ke^{\alpha(t-s)}\|g_+(s+t_n)-g_+(s)\|ds \leq$$

$$\leq \int_{t}^{d} K\xi e^{\alpha(t-s)}ds + \int_{d}^{\infty} 2M_gKe^{\alpha(t-s)}ds < \frac{K\xi}{\alpha} + \frac{2M_gK}{\alpha}e^{-\alpha(d-b)} \leq \frac{\varepsilon}{2}.$$

Thus, for sufficiently large n it is true that:

$$\|\varphi(t+t_n)-\varphi(t)\| \leq \|\varphi_+(t+t_n)-\varphi_+(t)\| + \|\varphi_-(t+t_n)-\varphi_-(t)\| < \varepsilon$$

for $t \in [a,b]$.

Next, we will show that the existence of a sequence $\{\bar{u}_n\}, \bar{u}_n \to \infty$ as $n \to \infty$, and positive numbers $\bar{\varepsilon}_0, \delta$ such that $\|\varphi(t+t_n)-\varphi(t)\| \geq \bar{\varepsilon}_0$ for $t \in [\bar{u}_n-\delta, \bar{u}_n +$ $+\delta]$.

According to uniform continuity of $g(t)$, there exists a positive number $\bar{\delta}$, which does not depend on the sequences $\{t_n\}$ and $\{u_n\}$, such that the inequalities:

$$\|g(t+t_n)-g(t_n+u_n)\| \leq \frac{\varepsilon_0}{4\sqrt{p}},$$

and

$$\|g(t)-g(u_n)\| \leq \frac{\varepsilon_0}{4\sqrt{p}}$$

are valid for every $t \in \left[u_n-\bar{\delta}\,, u_n+\bar{\delta}\right]$ and $n \in \mathbb{N}$.

Fix an arbitrary natural number n, and suppose that $g(t) = (g_1(t), \dots, g_p(t))$, where each $g_k(t), k = 1,2, \dots, p$, is a real valued function. It can be verified that there exists an integer $j_n, 1 \le j_n \le p$, such that:

$$\left|g_{j_n}(t_n + u_n) - g_{j_n}(u_n)\right| \ge \frac{\varepsilon_0}{\sqrt{p}}.$$

Hence, we have

$$\left|g_{j_n}(t + u_n) - g_{j_n}(t)\right| \ge$$

$$\ge \left|g_{j_n}(t_n + u_n) - g_{j_n}(u_n)\right| - \left|g_{j_n}(t + u_n) - g_{j_n}(t_n + u_n)\right| -$$

$$-\left|g_{j_n}(t) - g_{j_n}(u_n)\right| \ge \frac{\varepsilon_0}{2\sqrt{p}} \tag{1.2.4}$$

for $t \in \left[u_n - \bar{\delta}, u_n + \bar{\delta}\right]$.

One can confirm that there exist numbers $s_1^n, s_2^n, \dots, s_p^n$ in the interval $\left[u_n - -\bar{\delta}, u_n + \bar{\delta}\right]$ such that the equation:

$$|| \int_{u_n-\bar{\delta}}^{u_n+\bar{\delta}} \big(g(s + t_n) - g(s)\big) ds \, || = 2\bar{\delta}[\sum_{i=1}^{p} (g_i(\, s_i^n + t_n) - g_i(s_i^n))^2]^{1/2}$$

is valid. Using inequality (1.2.4) we obtain that:

$$|| \int_{u_n-\bar{\delta}}^{u_n+\bar{\delta}} (g(s + t_n) - g(s)) ds \, || \ge 2\bar{\delta}\left|g_{j_n}(s_i^n + t_n) - g_{j_n}(s_{j_n}^n)\right| \ge \frac{\bar{\delta}\varepsilon_0}{\sqrt{p}}.$$

By means of the equation:

$$\varphi\big(t_n + u_n + \bar{\delta}\big) - \varphi\big(t_n + \bar{\delta}\big) = \varphi\big(t_n + u_n - \bar{\delta}\big) - \varphi\big(u_n - \bar{\delta}\big) +$$

$$+ \int_{u_n-\bar{\delta}}^{u_n+\bar{\delta}} A[\varphi(s + t_n) - \varphi(s)] ds + \int_{u_n-\bar{\delta}}^{u_n+\bar{\delta}} [g(s + t_n) - g(s)] ds.$$

One can obtain that:

$$\left\|\varphi\big(t_n + u_n + \bar{\delta}\big) - \varphi\big(t_n + \bar{\delta}\big)\right\| \ge$$

$$\geq \frac{\bar{\delta}\varepsilon_0}{\sqrt{p}} - (1 + 2\bar{\delta}\|A\|) \sup_{t\in[u_n - \bar{\delta}, u_n + \bar{\delta}]} \|\varphi(t + t_n) - \varphi(t)\|.$$

Thus, $\sup_{t\in[u_n - \bar{\delta}, u_n + \bar{\delta}]} \|\varphi(t + t_n) - \varphi(t)\| \geq \frac{\bar{\delta}\varepsilon_0}{2(1+\bar{\delta}\|A\|)\sqrt{p}}$.

Now, suppose that $\sup_{t\in[u_n - \bar{\delta}, u_n + \bar{\delta}]} \|\varphi(t + t_n) - \varphi(t)\| = \|\varphi(t_n + \bar{u}_n) - \varphi(\bar{u}_n)\|$ for some $\bar{u}_n \in [u_n - \bar{\delta}, u_n + \bar{\delta}]$, and let us denote:

$$\bar{\varepsilon}_0 = \frac{\bar{\delta}\varepsilon_0}{4(1 + \bar{\delta}\|A\|)\sqrt{p}}$$

and

$$\delta = \frac{\bar{\delta}\alpha\varepsilon_0}{8M_g(1 + \bar{\delta}\|A\|)(\alpha + 2K\|A\|)\sqrt{p}}.$$

If $t \in [\bar{u}_n - \bar{\delta}, \bar{u}_n + \bar{\delta}]$, then we have:

$$\|\varphi(t + t_n) - \varphi(t)\| \geq \|\varphi(t_n + \bar{u}_n) - \varphi(\bar{u}_n)\| - \int_{\bar{u}_n}^{t} ||A|| \big| |\varphi(s + t_n) - \varphi(s)| \big| ds -$$

$$-\int_{\bar{u}_n}^{t} ||g(s + t_n) - g(s)|| ds \geq \frac{\bar{\delta}\varepsilon_0}{2(1 + \bar{\delta}||A||)\sqrt{p}} - \frac{4\delta M_g K||A||}{\alpha} - 2\delta M_g = \bar{\varepsilon}_0.$$

Hence, $\|\varphi(t + t_n) - \varphi(t)\| \geq \bar{\varepsilon}_0$ for each t from the intervals $[\bar{u}_n - \bar{\delta}, \bar{u}_n + \bar{\delta}]$, $n \in \mathbb{N}$. One can confirm that the sequence $\{\bar{u}_n\}$ diverges to infinity. Consequently, $\varphi(t)$ is the unique unpredictable solution of system (1.2.1).

Example 1. It was shown in paper [3, p. 86] that the presence of an unpredictable function is inevitably accompanied with Poincare chaos. Therefore, we can look for a confirmation of the results for unpredictability observing irregularity in simulations. The approach is effective for asymptotically stable unpredictable solutions, and it is just illustrative for hyperbolic systems with unstable solutions. In the latter case we rely on the fact that any solution becomes unpredictable ultimately.

Consider the system:

$$\begin{cases} x_1' = -2x_1 + 2x_2 - 50\Theta(\mathrm{t}) \\ x_2' = x_1 - 3x_2 - 5\Theta^3(t), \end{cases} \tag{1.2.5}$$

where $\Theta(t) = \int_{-\infty}^{t} e^{-2.5(t-s)}\Omega(s)ds$, is the unpredictable function defined in section 1. The eigenvalues of the matrix of coefficients of system (1.2.5) are −2 and −0.5. One can confirm that the perturbation function $(-50\Theta(t), -5\Theta^3(t))$ is unpredictable in accordance with Properties 1 and 3. By Theorem 1.2.1 there is an asymptotically stable unpredictable solution $(x_1(t), x_2(t))$ of system (1.2.5). Consequently, any solution of the equation behaves irregularly ultimately. This is seen from the simulation of the solution with $x_1(0) = 0{,}18, x_2(0) = 0{,}01$ in figure 3.

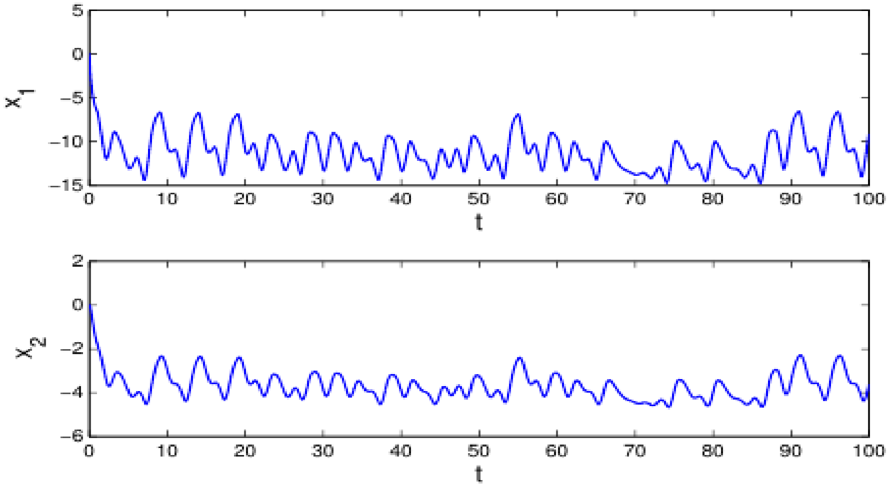

Figure 3 – The time series of the x_1 and x_2 coordinates of system (2.1.8) with the initial conditions $x_1(0) = 0{,}18, x_2(0) = 0{,}01$. The figure manifests the irregular behavior of the solution

The next example is devoted to a system of differential equations whose matrix of coefficients admit both positive and negative eigenvalues.

Example 2. Let us take into account the system:

$$\begin{cases} y_1' = -1000y_1 + 0.23y_2 + 120x_2^3(t) + 160 \\ y_2' = 6y_1 + 0.000002y_2 - 0.1x_1(t) + 20, \end{cases} \tag{1.2.6}$$

where $(x_1(t), x_2(t))$ is the solution of (1.2.5) depicted in figure 1. The eigenvalues of the matrix of coefficients of system (1.2.6) are -1000 and 0.00138. The perturbation function $(120x_2^3(t) + 160, -0.1x_1(t) + 20)$ is unpredictable. According to the Theorem 1.2.1, system (1.2.6) possesses a unique unpredictable solution. The simulation results for system (1.2.6) corresponding to the initial conditions $y_1(0) = 0$ and $y_2(0) = 0.1$ are shown in figure 4. The time series of both y_1 and y_2 coordinates in the figure confirm the presence of irregularity in the dynamics of system (1.2.6).

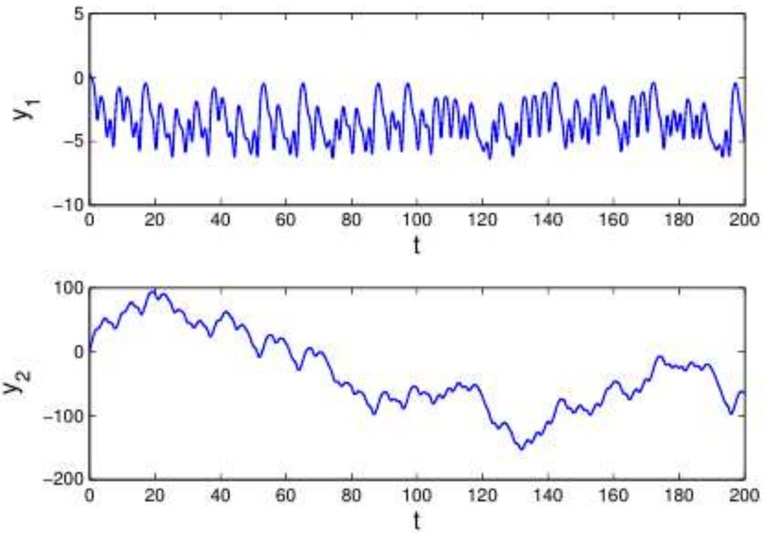

Figure 4 – The time series for the y_1 and y_2 coordinates of system (1.2.6) with the initial conditions $y_1(0) = 0, y_2(0) = 0.1$. The irregular behavior of the solution reveals the presence of an unpredictable solution in the dynamics of (1.2.6).

1.3 Hyperbolic quasilinear systems

The main object of the present section is the system of quasilinear differential equations:

$$x'(t) = Ax(t) + f(x(t)) + g(t), \tag{1.3.1}$$

where $x(t) \in \mathbb{R}^p$, p is a fixed natural number, the function $f: \mathbb{R}^p \to \mathbb{R}^p$ is continuous in all of its arguments, $f(0,0,\ldots,0) = (0,0,\ldots,0)$, and all eigenvalues of the constant matrix $A \in \mathbb{R}^{p\times p}$ have nonzero real parts. We assume that $\Re e(\lambda_i) < 0$, $i = 1,2,\ldots,q$, and $\Re e(\lambda_i) > 0$, $i = q+1,\ldots,p$, where $1 \le q < p, \lambda_i, i = 1,2,\ldots,p$, are the eigenvalues of the matrix A, and $\Re e(\lambda_i)$ denotes the real part of the eigenvalue λ_i. Moreover, the function $g: \mathbb{R} \to \mathbb{R}^p$ is unpredictable with positive numbers ε_0, δ and sequences $\{t_n\}, \{u_n\}$ determined in Definition 1.1.1. Our purpose is to prove that system (1.3.1) possesses a unique unpredictable solution, provided that the function $g(t)$ is unpredictable and the solution is uniformly exponentially stable if all eigenvalues of the matrix A have negative real parts.

The following condition on system (1.3.1) is required.

$(C1)$ There exists a positive number L_f such that $\|f(x_1) - f(x_2)\| \le$ $\le L_f\|x_1 - x_2\|$ for all $x_1, x_2 \in \mathbb{R}^p$.

It is known that one can find a regular $p \times p$ matrix B such that the transformation $x = By$ reduces system (1.3.1) to the system:

$$y'(t) = Cy(t) + F(y) + G(t), \tag{1.3.2}$$

where $C = B^{-1}AB, F(y) = B^{-1}f(By)$, and $G(t) = B^{-1}g(t)$. In system (1.3.2), the matrix C is of the form $\mathrm{diag}(C_-, C_+)$, where the eigenvalues of the $q \times q$ matrix C_-

and $(p-q)\times(p-q)$ matrix C_+ respectively have negative and positive real parts. Let us denote $F(y)=(F_\_(y),F_+(y))$ and $G(t)=(G_\_(t),G_+(t))$, where the vector-functions $F_\_$ and $G_\_$ are of dimension q and the vector-functions F_+ and G_+ are of dimension *p-q*.

It can be verified that $F(0,0,\ldots,0)=(0,0,\ldots 0)\in\mathbb{R}^p$ the function F is Lipschitzian with the Lipschitz constant:

$$L_F=||B||\,||B^{-1}||L_f,$$

under the condition (C1), that is, $\left||F(y_1)-F(y_2)|\right|\le L_F\left||y_1-y_2|\right|$ for all $y_1,y_2\in$ $\in\mathbb{R}^p$. In addition, according to the Lemma 1.2.1 from the previous section, the function $G(t)$ is unpredictable.

Fix a number $H>0$ such that $||g(t)||<H$. Then $||G(t)||<||B^{-1}||H.$

Next the main result is formulated for system (1.3.2) and interpreted for system (1.3.1).

One can confirm that there exist numbers $K\ge 1$ and $\alpha>0$ such that $\left||e^{C_t}|\right|\le$ $\le Ke^{-\alpha t}$ for all $t\ge 0$ and $||e^{C_+t}||\le Ke^{\alpha t}$ for all $t\le 0$.

The following conditions are needed.

$(C2)$ $\frac{2}{\alpha}K(||B||||B^{-1}||L_f+1)<1.$

According to the theory of differential equations [87, p. 150], a function $\varphi(t)=(\varphi_\_(t),\varphi_+(t))$ which is bounded on the whole real axis is a solution of system (1.3.2) if only if it satisfies the equations:

$$\varphi_\_(t)=\int_{-\infty}^{t}e^{C_\_(t-s)}[F_\_(\varphi(s))+G_\_(s)]ds,$$
$$\varphi_+(t)=-\int_{t}^{\infty}e^{C_+(t-s)}[F_+(\varphi(s))+G_+(s)]ds. \tag{1.3.3}$$

Let U be the set of all uniformly continuous and bounded functions $\psi(t):\mathbb{R}\to\mathbb{R}^p$ such that $||\psi||_1\le||B^{-1}||H$, where the norm $||.||_1$ is defined by $||\psi||_1=\sup\limits_{t\in\mathbb{R}}||\psi(t)||$ and $\psi(t+t_n)\to\psi(t)$ and $n\to\infty$ uniformly on each compact subset of $\mathbb{R}$.

Lemma 1.3.2. Suppose that the conditions (C1) –(C2) are valid then the system (1.3.2) possesses a unique solution $\omega(t)\in U$ such that $\sup\limits_{t\in\mathbb{R}}||\omega(t)||\le||B^{-1}||H$.

Proof. In the proof, we will show that system (1.3.2) possesses a unique Poisson stable solution by using the contraction mapping principle, and this implies the existence of a unique Poisson stable solution of system (1.3.1).

Since the function $g(t)$ is Poisson stable, there exists a sequence $\{t_n\}, t_n \to \infty$ as $n \to \infty$, such that $g(t+t_n) \to g(t)$ as $n \to \infty$ uniformly on each compact subset of $\mathbb{R}$.

Define an operator Π on U through the equations:

$$\Pi_{-}\psi(t) = \int_{-\infty}^{t} e^{C_{-}(t-s)}[F_{-}(\psi(s)) + G_{-}(s)]ds$$

and

$$\Pi_{+}\psi(t) = -\int_{t}^{\infty} e^{C_{+}(t-s)}\left[F_{+}\left(\psi(s)\right) + G_{+}(s)\right]ds,$$

such that $\Pi\psi(t) = (\Pi_{-}\psi(t), \Pi_{+}\psi(t))$.

Fix an arbitrary function $\psi(t)$ that belongs to U. For all $t \in \mathbb{R}$, we have:

$$\left||\Pi_{-}\psi(t)|\right| \le \int_{-\infty}^{t} ||e^{C_{-}(t-s)}||||F_{-}(\psi(s)) + G_{-}(s)||ds \le$$

$$\le \int_{-\infty}^{t} K\left(L_F\left||\psi(s)|\right| + \left||B^{-1}|\right|H\right)e^{-\alpha(t-s)}ds \le \frac{1}{\alpha}K\left||B^{-1}|\right|H\left(\left||B|\right|\left||B^{-1}|\right|L_f + 1\right)$$

and

$$\left||\Pi_{+}\psi(t)|\right| \le \int_{t}^{\infty} \left||e^{C_{+}(t-s)}|\right| \left||F_{+}\left(\psi(s)\right) + G_{+}(s)|\right| ds \le$$

$$\le \int_{t}^{\infty} K\left(L_F\left||\psi(s)|\right| + \left||B^{-1}|\right|H\right)e^{-\alpha(t-s)}ds \le \frac{1}{\alpha}K\left||B^{-1}|\right|H\left(\left||B|\right|\left||B^{-1}|\right|L_f + 1\right).$$

Therefore, by condition $(C2)$ $||\Pi\psi||_1 \le ||B^{-1}||H$.

Now, let us fix an arbitrary positive number ε and a compact subset $[a, b]$ of $\mathbb{R}$, where $b > a$. Suppose that a_0 and b_0 are numbers satisfying $a_0 < a, b < b_0$ such that the inequalities:

$$\frac{2}{\alpha}K\left||B^{-1}|\right|H\left(\left||B|\right|\left||B^{-1}|\right|L_f + 1\right)e^{-\alpha(a-a_0)} < \frac{\varepsilon}{4}, \qquad (1.3.4)$$

$$\frac{2}{\alpha}K||B^{-1}||H(||B||L_f+1)e^{\alpha(b-b_0)}<\frac{\varepsilon}{4}, \qquad (1.3.5)$$

are valid, and let γ be a number such that:

$$\gamma>\frac{4K||B^{-1}||(||B||L_f+1)}{\alpha}. \qquad (1.3.6)$$

There exists a natural number n_0 such that if $n \geq n_0$, then $||G(t+t_n)+$ $+G(t)||<\frac{\varepsilon}{\gamma}$ and $||\psi(t+t_n)-\psi(t)||<\frac{\varepsilon}{\gamma}$ for all $t \in [a_0,b_0]$.

Therefore, one can verify for $n \geq n_0$, and $t \in [a_0,b_0]$ that:

$$||\Pi\_\psi(t+t_n)-\Pi\_\psi(t)|| \leq \int_{-\infty}^{a_0} ||e^{C\_(t-s)}||(||F\_(\psi(s+t_n))-F\_(\psi(s))||+$$

$$+||G\_(s+t_n)-G\_(s)||)ds+\int_{a_0}^{t} ||e^{C\_(t-s)}||(||F\_(\psi(s+t_n))-F\_(\psi(s))||+$$

$$+||G\_(s+t_n)-G\_(s)||)ds \leq$$

$$\leq \frac{2}{\alpha}K\big(||B||||B^{-1}||L_f||B^{-1}||H+||B^{-1}H||\big)e^{-\alpha(a-a_0)}+$$

$$+\frac{\varepsilon}{\alpha\gamma}K\big(||B||||B^{-1}||L_f+||B^{-1}||\big)(1-e^{-\alpha(t-a_0)})$$

and

$$\big||\Pi_+\psi(t+t_n)-\Pi_+\psi(t)|\big| \leq \int_{b_0}^{\infty} ||e^{C_+(t-s)}||(||F_+(\psi(s+t_n))-F_+(\psi(s))||+$$

$$+||G_+(s+t_n)-G_+(s)||)ds+\int_{t}^{b_0} ||e^{C_+(t-s)}||(||F_+(\psi(s+t_n))-F_+(\psi(s))||+$$

$$+||G_+(s+t_n)-G_+(s)||)ds \leq$$

$$\leq \frac{2}{\alpha}K\big(||B||||B^{-1}||L_f||B^{-1}||H+||B^{-1}H||\big)e^{-\alpha(t-b_0)}+$$

$$+\frac{\varepsilon}{\alpha\gamma}K\big(||B||||B^{-1}||L_f+||B^{-1}||\big)(1-e^{-\alpha(t-b)}).$$

Now, using the inequalities (1.3.4) to (1.3.6), one can obtain for $t \in [a,b]$ that

$$||\Pi\_\psi(t+t_n)-\Pi\_\psi(t)||<\frac{\varepsilon}{2}$$

and

$$||\Pi_+\psi(t+t_n)-\Pi_+\psi(t)||<\frac{\varepsilon}{2}$$

provided that $n \geq n_0$. Hence, if $n \geq n_0$, then the inequality $||\Pi\psi(t+t_n)-$ $-\Pi\psi(t)||<\varepsilon$ is valid for all $t\in[a,b]$, and therefore $\Pi\psi(t+t_n)\to\Pi\psi(t)$ uniformly as $n\to\infty$ on each compact subset of $\mathbb{R}$.

Additionally, it can be shown that $\Pi\psi(t)$ is uniformly continuous since its derivative is bounded.

Next, we will show that the operator $\Pi \to U$ is contractive. Let $\psi_1(t)$ and $\psi_2(t)$ be functions in U. Then, we have that

$$||\Pi\_\psi_1(t)-\Pi\_\psi_2(t)||\leq\int_{-\infty}^{t}||e^{C\_(t-s)}||||F\_(\psi_1(s))-F\_(\psi_2(s))||ds\leq$$

$$\leq\frac{K||B||||B^{-1}||L_f}{\alpha}||\psi_1-\psi_2||_0$$

and

$$||\Pi_+\psi_1(t)-\Pi_+\psi_2(t)||\leq\int_{t}^{\infty}||e^{C_+(t-s)}||||F_+(\psi_1(s))-F_+(\psi_2(s))||ds\leq$$

$$\leq\frac{K||B||||B^{-1}||L_f}{\alpha}||\psi_1-\psi_2||_0.$$

Therefore, the inequality

$$||\Pi\psi_1(t)-\Pi\psi_2(t)||\leq\frac{K||B||||B^{-1}||L_f}{\alpha}||\psi_1-\psi_2||_0$$

holds, and according to the condition $(C2)$ the operator $\Pi\colon U\to U$ is contractive.

By contraction theorem there exists fixed point of the operator Π which is a bonded solution $\omega(t)$ of the system (1.3.2) and it satisfies an inequality $\sup_{t\in\mathbb{R}}||\omega(t)|| \le ||B^{-1}||H$.

Now we will show that the sequence $\omega(t+t_n)$ converges on any compact interval on $\mathbb{R}$. For this reason, consider the sequence of approximations $\omega_k(t),\ k = 0,1,\ldots$, such that $\omega_0(t) = G(t)$ and $\omega_{k+1}(t) = \Pi\omega_k(t)$. This sequence uniformly converges on $\mathbb{R}$ to the function $\omega(t)$ and each function $\omega_k(t)$ satisfies the condition that $\omega_k(t+t_n)$ converges uniformly on any compact subinterval of the real axis. Thus, we have that:

$$|\omega(t+t_n)-\omega(t)| \le |\omega(t+t_n)-\omega_k(t+t_n)| + |\omega_k(t+t_n)-\omega_k(t)| +$$

$$+|\omega_k(t)-\omega(t)| \le \varepsilon,$$

for a fixed positive ε with sufficiently large k and n. This is why, $\omega(t+t_n)$, $n = 1,2,\ldots$, converges to $\omega(t)$ uniformly on each compact interval of the real axis. The lemma is proved.

The next theorem is concerned with the unpredictable solution of system (1.3.1).

Theorem 1.3.1. Suppose that conditions $(\mathrm{C1})-(\mathrm{C2})$ are valid, then the system (1.3.1) possesses a unique unpredictable solution. Moreover, the unpredictable solution is uniformly exponentially stable if all eigenvalues of the matrix A have negative real parts.

Proof. According to Lemma 1.3.2, system (1.3.1) possesses a unique Poisson stable solution $\omega(t)$. Therefore, to prove that system (1.3.1) admits a unique unpredictable solution, it remains to show that $\omega(t)$ satisfies the unpredictable property.

We will show the existence of a sequence $\bar{u}_n, \bar{u}_n \to \infty$ as $n \to \infty$, and positive numbers $\bar{\varepsilon}_0, \bar{\delta}$ such that $\left||\omega(t+t_n)-\omega(t_n)|\right| \ge \bar{\varepsilon}_0$ for $t \in [\bar{u}_n - \bar{\delta}, \bar{u}_n + \bar{\delta}]$.

One can find a positive number κ and natural numbers l, k and $j = 1,\ldots,p$, such that

$$\kappa < \delta, \tag{1.3.7}$$

$$\kappa(1/2 - (\frac{1}{l} + \frac{2}{\kappa})(L + ||A||)) > \frac{3}{2l}\,, \tag{1.3.8}$$

$$||\omega(t+s)-\omega(t)|| < \bar{\varepsilon}_0\, min\,(\tfrac{1}{k} + \tfrac{2}{4l}), t \in \mathbb{R}, |s| < \kappa. \tag{1.3.9}$$

Let the numbers κ, l and k as well as a number $n \in \mathbb{N}$, be fixed. Denote $\Delta = ||\omega(t_n + u_n) - \omega(u_n)||$ and consider two cases (i)$\Delta \ge \varepsilon_0/l$ and (ii)$\Delta < \varepsilon_0/l$.

(i) By (1.3.9) we have that:

$\left||\omega(t+t_n)-\omega(t)|\right| \ge \left||\omega(t_n+u_n)-\omega(u_n)|\right| - \left||\omega(u_n)-\omega(t)|\right| -$

$$-||\omega(t+t_n)-\omega(t_n+u_n)|| \ge \frac{\varepsilon_0}{l}-\frac{\varepsilon_0}{4l}-\frac{\varepsilon_0}{4l}=\frac{\varepsilon}{2l}, \qquad (1.3.10)$$

if $t \in [u_n-\kappa, u_n+\kappa]$ and $n \in \mathbb{N}$.

(ii) One can find that from (1.3.9) it follows that

$$\left||\omega(t+t_n)-\omega(t)|\right| < \frac{\varepsilon_0}{l}+\frac{\varepsilon_0}{k}+\frac{\varepsilon_0}{k}=\varepsilon_0(\frac{1}{l}+\frac{2}{k}), \qquad (1.3.11)$$

if $t \in [u_n, u_n+\kappa]$.

It is true that

$$\omega(t+t_n)-\omega(t)= \omega(t_n+u_n)-\omega(u_n)+\int_{u_n}^{t} A[\omega(s+t_n)-\omega(s)]ds+$$

$$+\int_{u_n}^{t}[f(\omega(s+t_n))-f(\omega(s))]ds+\int_{u_n}^{t}[g(s+t_n)-g(s)]ds. \qquad (1.3.12)$$

We obtain from (1.3.7) -(1.3.9) and (1.3.11) that

$$\left|\omega_j(t+t_n)-\omega_j(t)\right| \ge \int_{u_n}^{t}\left|g_j(s+t_n)-g_j(s)\right|ds -$$

$$-\int_{u_n}^{t}|\sum_{j=1}^{p} a_{ij}\,[\omega_i(s+t_n)-\omega_i(s)]|ds-\int_{u_n}^{t}\left|f_i\big(\omega(s+t_n)\big)-f_i\big(\omega(s)\big)\right|ds-$$

$$-|\omega_j(t_n+u_n)-\omega_j(u_n)| \ge \frac{\kappa}{2}\varepsilon_0-\kappa||A||\varepsilon_0(\frac{1}{l}+\frac{2}{k})-\kappa L\varepsilon_0(\frac{1}{l}+\frac{2}{k})-\frac{\varepsilon_0}{l} \ge \frac{\varepsilon_0}{2l},$$

for $t \in [u_n+\kappa/2, u_n+\kappa]$.

Thus, $||\omega(t+t_n)-\omega(t)|| \ge \bar{\varepsilon}_0=\frac{\varepsilon_0}{2l}$ for each t from the intervals $[\bar{u}_n-\bar{\delta}, \bar{u}_n+\bar{\delta}]$, where $\bar{u}_n=u_n+\frac{3\kappa}{4}, \bar{\delta}=\frac{\kappa}{4}, n \in \mathbb{N}$. Consequently, the bounded solution $\omega(t)$ is unpredictable.

On the other hand, one can show in a similar way to the proof of Theorem 4.1 [2, p. 257] that if all eigenvalues of the matrix A have negative real parts, then the unpredictable solution of system (1.3.1) is uniformly exponentially stable under condition $(C3)$.

Example 3. Consider the system

$$\begin{cases} x_1' = -3x_1 + 2x_2 - 0.1x_2^3 + 20\Theta(t) + 3 \\ x_2' = x_1 - 2x_2 + 0.5x_1^2 - 15\Theta(t) - 2, \end{cases} \qquad (1.3.13)$$

where $\Theta(t) = \int_{-\infty}^{t} e^{-2(t-s)}\Omega(s)ds$, is the unpredictable function defined in section 1. The eigenvalues of the matrix of coefficients of system (1.3.13) are −1 and −4. By the properties of unpredictable function, it can be confirmed that the perturbation function $(20\Theta(t) + 3, -15\Theta(t) - 2)$ is unpredictable. According to Theorem 1.3.1, there is a unique exponentially stable unpredictable solution $(x_1(t), x_2(t))$ of system (1.3.13). Figures 5 and 6 present the function $(t) = (\psi_1(t), \psi_2(t))$, which exponentially tends to the solution $x(t)$.

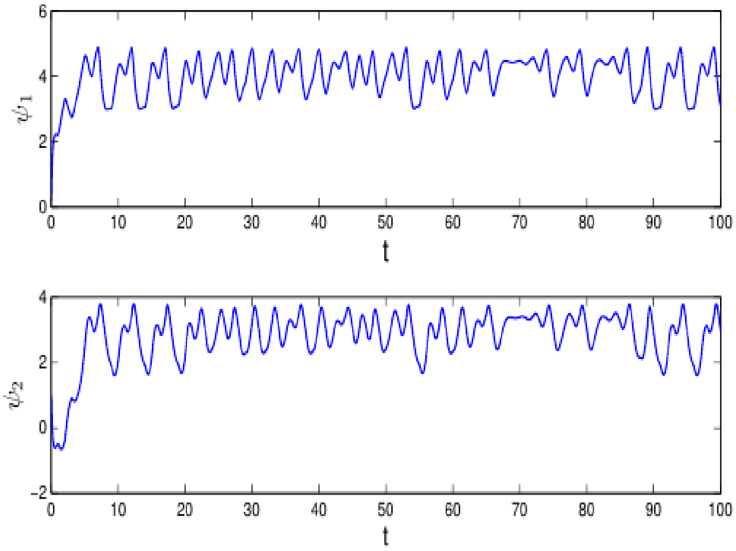

Figure 5 – The time series of the $\psi_1(t), \psi_2(t)$ coordinates with the initial value $\psi_1(0) = 1.18$, $\psi_2(0) = 1.01$. The figure manifests the irregular behavior of the system

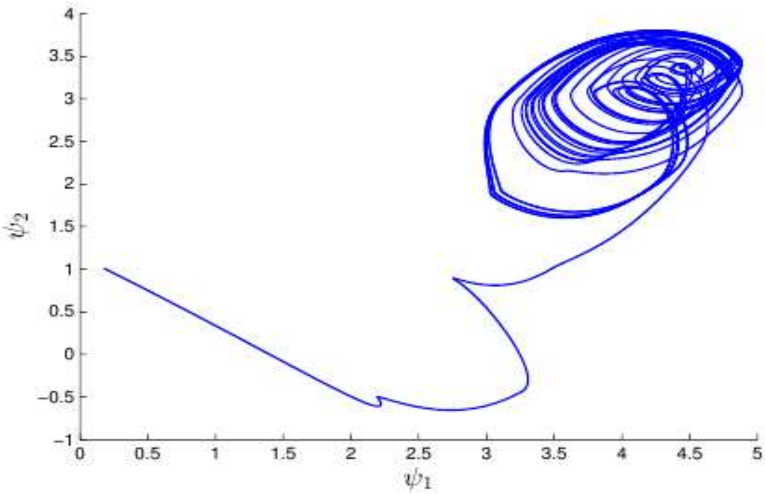

Figure 6 – The irregular trajectory of the function $\psi(t)$

*Example 4***.** Let us consider the system

$$\begin{cases} y_1' = 4y_1 - 0.021y_2^3 - 2x_2^3(t) \\ y_2' = 7y_1 - 6y_2 - 0.014y_1^2 + 3x_1^3(t), \end{cases} \qquad (1.3.14)$$

where $(x_1(t), x_2(t))$ is the unpredictable solution of system (1.3.13). The eigenvalues of the matrix of coefficients of system (1.3.14) are -6 and 4. The function $x(t) = = (-2x_2^3(t), 3x_1^3(t))$ is unpredictable, and therefore, the system possesses a unique unpredictable solution by Theorem 1.3.1. Figures 7 and 8 show the simulation results for the function $\phi(t)$, which approximates the solution $y(t)$ of the system (1.3.14).

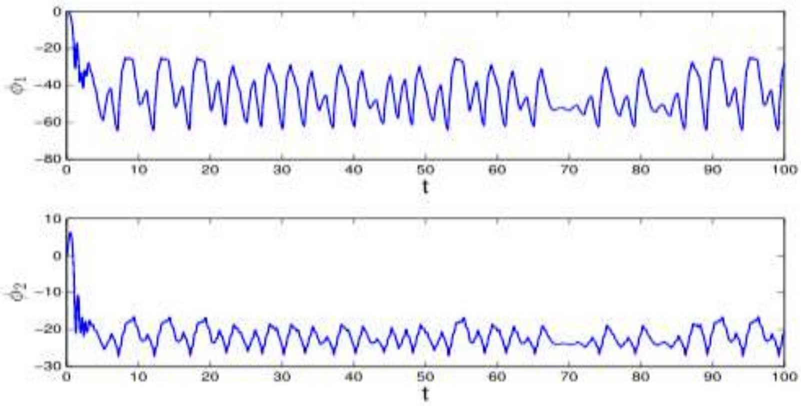

Figure 7 - The coordinates of the function $\phi(t)$ with initial conditions $\phi_1(0) = 0$, $\phi_2(0) = 0$. The behavior of the function reveals the presence of an unpredictable solution in the dynamics of (1.3.14)

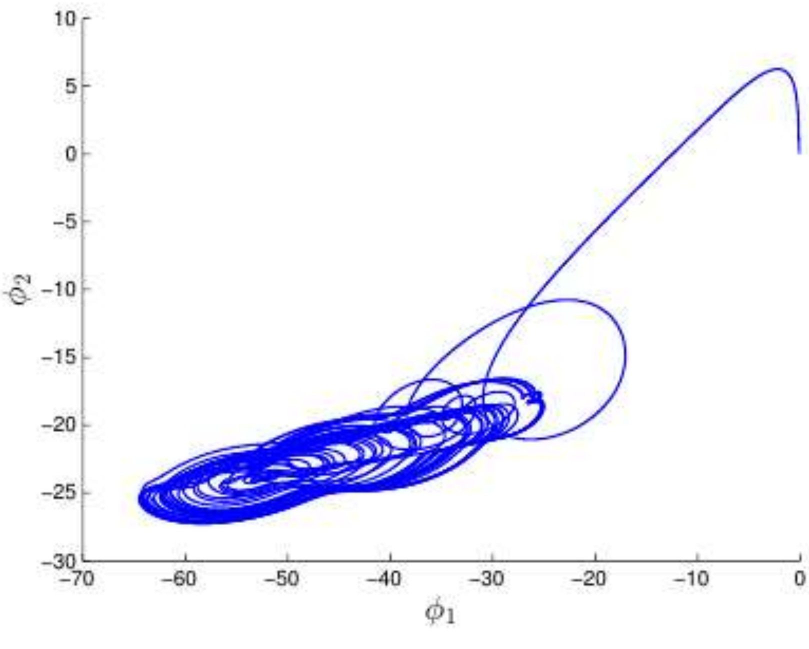

Figure 8 - The trajectory of the function $\phi(t)$. One can observe the irregular behavior in the graph

1.4 Strongly unpredictable solutions of differential equations

Consider the following differential equation:

$$x'(t) = Ax(t) + f(t,x), \tag{1.4.1}$$

where $t \in \mathbb{R}, x \in \mathbb{R}^p$, p is a fixed natural number, all eigenvalues of the constant matrix $A \in \mathbb{R}^{p\times p}$ have negative real parts, $f: \mathbb{R} \times G \to \mathbb{R}^p, f = (f_1, f_2, \dots, f_p)$, $G = \{x \in \mathbb{R}^p, ||x||_1 < H\}$, where H is a positive number. It is true that there exist two real numbers $K \geq 1$ and $\gamma < 0$ such that $||e^{At}|| \leq Ke^{\gamma t}$ for all $t \geq 0$.

One can see that the main difference between system (1.3.1) and system (1.4.1) is that the perturbation in the former one is less general than that of the latter one.

Assume that the following conditions are valid:

(C1) the function $f(t,x)$ is strongly unpredictable in the sence of Definition 1.1.4, so that there exists a positive number M such that $\sup\limits_{\mathbb{R}\times G} ||f(t,x)|| = M < \infty$;

(C2) there exists a positive constant L such that the function $f(t,x)$ satisfies the inequality $||f(t,x_1) - f(t,x_2)|| \leq L||x_1 - x_2||$ for all $t \in \mathbb{R}, x_1, x_2 \in G$;

(C3) $\gamma < -\frac{KM}{H}$;

(C4) $\gamma < -KL$.

Our purpose is to prove that the system (1.4.1) possesses a unique strongly unpredictable solution, provided that the function $f(t,x)$ is strongly unpredictable in *t*. Moreover, we prove that the solution is uniformly globally exponentially stable. Additionally, existence of an unpredictable solution, which is not strongly unpredictable, is considered for the system (1.4.1).

Let U be the set of all uniformly continuous functions $\psi(t) = (\psi_1, \psi_2, \dots, \psi_p)$, $\psi_i \in \mathbb{R}, i = 1,2,\dots,p$, such that $||\psi||_1 < H$, and $\psi(t + t_n) \to \psi(t)$ as $n \to \infty$ uniformly on each closed and bounded interval of the real axis, where sequence t_n is the same as for function $f(t,x)$ in system (1.4.1).

According to the theory of differential equations [88], a function $\omega(t) = (\omega_1, \omega_2, \dots, \omega_p)$ bounded on the whole real axis is a solution of system $f(t,x)$ if only if the integral equation:

$$\omega(t) = \int_{-\infty}^{t} e^{A(t-s)} f(s, \omega(s)) ds, t \in \mathbb{R},$$

is satisfied.

Lemma 1.4.1. Suppose that conditions (C1) -(C4) are valid, then the system (1.4.1) possesses a unique solution ω(t) ∈ U.

Proof. Define an operator Π on U such that:

$$\Pi\psi(t) = \int_{-\infty}^{t} e^{A(t-s)} f(s,\psi(s))ds, t \in \mathbb{R}. \quad (1.4.2)$$

Fix an arbitrary function $\psi(t)$ that belongs to U. We have that:

$$||\Pi\psi(t)|| = \int_{-\infty}^{t} ||e^{A(t-s)}|| \, ||f(s,\psi(s))||ds \leq \frac{KM}{|\gamma|}$$

for all $t \in \mathbb{R}$. Therefore, by condition (C3) it is true that $||\Pi\psi||_1 < H$.

Fix an arbitrary positive ε and a closed interval $[a,b], -\infty < a < b < \infty$, of the real axis. We will show that for sufficiently large n it is true that $||\Pi\psi(t+t_n) - -\Pi\psi(t)|| < \varepsilon$ on $[a,b]$. Let us choose two numbers $c < a$, and $\xi > 0$ satisfying

$$\frac{2KM}{|\gamma|} e^{\gamma(a-c)} < \frac{\varepsilon}{2} \quad (1.4.3)$$

and

$$\frac{K}{|\gamma|}\xi(L+1)[1-e^{\gamma(b-c)}] < \frac{\varepsilon}{2}. \quad (1.4.4)$$

Theorem 1.4.1. If conditions (C1) -(C4) are fulfilled, then the system (1.4.1) admits a unique uniformly exponentially stable strongly unpredictable solution.

Proof. According to Lemma 1.4.1, system (1.4.1) has a unique solution $\omega(t) \in U$. What is left is to verify that the unpredictability property is valid. One can find a positive number κ, natural numbers l, k such that for all $j = 1,\ldots,p$ the following inequalities are valid,

$$\kappa < \delta, \quad (1.4.5)$$

$$\kappa\left(\frac{1}{2} - (\frac{1}{l} + \frac{2}{k})(L + ||A||)\right) > \frac{3}{2l}, \quad (1.4.6)$$

$$||\omega(t+s) - \omega(t)|| < \varepsilon_0 min\left(\frac{1}{k}, \frac{1}{4l}\right), \quad t \in \mathbb{R}, \quad |s| < \kappa. \quad (1.4.7)$$

Let the numbers κ, l, k and j as well as $n \in \mathbb{N}$, be fixed.

Denote $\Delta = \left|\omega_j(u_n + t_n) - \omega_j(u_n)\right|$ and consider two cases: (i) $\Delta \geq \varepsilon_0/l$, (ii) $\Delta < \varepsilon_0/l$ such the remaining proof falls into two parts.

(i) We have that:

$$\left|\omega_j(t+t_n)-\omega_j(t)\right| \geq |\omega_j(t_n+u_n)-\omega_j(u_n)| - |\omega_j(u_n)-\omega_j(t)| -$$

$$-|\omega_j(t+t_n)-\omega_j(t_n+u_n)| \geq \frac{\varepsilon_0}{l}-\frac{\varepsilon_0}{4l}-\frac{\varepsilon_0}{4l}=\frac{\varepsilon}{2l}, \qquad (1.4.8)$$

if $t \in [u_n-\kappa, u_n+\kappa]$ and $n \in \mathbb{N}$.

(ii) It can easily find that (1.4.7) implies

$$||\omega(t+t_n)-\omega(t)|| < \frac{\varepsilon_0}{l}+\frac{\varepsilon_0}{k}+\frac{\varepsilon_0}{k}=\varepsilon_0\left(\frac{1}{l}+\frac{2}{k}\right) \qquad (1.4.9)$$

if $t \in [u_n, u_n+\kappa]$.

From the last inequality and (1.4.5)- (1.4.7) it follows that

$$\left|\omega_j(t+t_n)-\omega_j(t)\right| \geq \int_{u_n}^{t} |f_j(s+t_n,\omega(s+t_n))-f_j(s,\omega(s+t_n))|ds -$$

$$-\int_{u_n}^{t} |f_j(s,\omega(s+t_n))-f_j(s,\omega(s))|ds - \int_{u_n}^{t} |\sum_{j=1}^{p} a_{ij}[\omega_j(s+t_n)-\omega_j(s)]|ds -$$

$$-|\omega_j(t_n+u_n)-\omega_j(u_n)| \geq \frac{\kappa}{2}\varepsilon_0-\kappa L\varepsilon_0(\frac{1}{l}+\frac{2}{k})-\kappa||A||\varepsilon_0(\frac{1}{l}+\frac{2}{k})-\frac{\varepsilon_0}{l} \geq \frac{\varepsilon_0}{2l})$$

for $t \in [u_n+\kappa/2, u_n+\kappa]$.

Thus, the solution $\omega(t)$ is strongly unpredictable with $\bar{u}_n = u_n + \frac{3\kappa}{4}, \bar{\sigma} = \frac{\kappa}{4}$.

The uniformly exponentially stability of the solution $\omega(t)$ can be verified as stability of a bounded solution in [87, p. 100; 88, p. 40]. The theorem is proved.

We have considered the problem of existence and uniqueness of strongly unpredictable solutions. In what follows, we will search for quasilinear systems with unpredictable solutions, which are not strongly unpredictable. For this reason, assume that the following condition is valid.

(C5) The function $f(t,x)$ is unpredictable in the sense of Definition 1.1.3.

Theorem 1.4.2. Suppose that the conditions (C2)-(C5) hold. Then the system (1.4.1) admits a unique uniformly exponentially stable unpredictable solution.

Proof. One can easily see, proceeding in the way of the last theorem, that there exists a unique solution $\omega(t) \in U$ for system (1.4.1). The solution is uniformly asymptotically stable. What is left is to show that the unpredictability property is valid.

One can find a positive number κ and natural numbers l, k and $j = 1, \ldots, p$, such that:

$$\kappa < \sigma \qquad (1.4.10)$$

$$|f_j(t_n+u_n+s,x)-f_j(u_n+s,x)| \geq \varepsilon_0/2, ||x||<H, |s|<\kappa, n \in \mathbb{N}, \qquad (1.4.11)$$

$$\kappa\left(1/4 \;-\; (\frac{1}{l}+\frac{2}{k})(L+||A||)\right) > \frac{3}{2l}\,, \qquad (1.4.12)$$

$$||\omega(t+s)-\omega(t)|| < \varepsilon_0 min\left(\frac{1}{k},\frac{1}{4l}\right), t \in \mathbb{R}, |s|<\kappa\,. \qquad (1.4.13)$$

Let the numbers $\kappa, l,$ and k as well as a number $n \in \mathbb{N}$, be fixed. Denote $\Delta=$ $= \left||\omega(t_n+u_n)-\omega(u_n)|\right|$ and consider two alternave cases (i) $\Delta \geq \varepsilon_0/l$, (ii) $\Delta < \varepsilon_0/l$.

(i) By (1.4.13) we have that:

$$\left||\omega(t+t_n)-\omega(t)|\right| \geq \left||\omega(t_n+u_n)-\omega(u_n)|\right| - \left||\omega(u_n)-\omega(t)|\right| -$$

$$-||\omega(t+t_n)-\omega(t_n+u_n)|| \geq \frac{\varepsilon_0}{l} - \frac{\varepsilon_0}{4l} - \frac{\varepsilon_0}{4l} = \frac{\varepsilon}{2l}, \qquad (1.4.14)$$

if $t \in [u_n-\kappa, u_n+\kappa]$ and $n \in \mathbb{N}$.

(ii) One can find that from (1.4.13) it follows that:

$$||\omega(t+t_n)-\omega(t)|| < \frac{\varepsilon_0}{l} + \frac{\varepsilon_0}{k} + \frac{\varepsilon_0}{k} = \varepsilon_0(\frac{1}{l}+\frac{2}{k}) \qquad (1.4.15)$$

if $t \in [u_n, u_n+\kappa]$.

It is true that

$$\omega(t+t_n)-\omega(t) = \omega(t_n+u_n)-\omega(u_n) + \int_{u_n}^{t} A[\omega(s+t_n)-\omega(s)]ds +$$

$$+\int_{u_n}^{t} [f(s+t_n,\omega(s+t_n)) - f(s,\omega(s))|ds, t \in \mathbb{R}. \qquad (1.4.16)$$

We obtain from (1.4.10) -(1.4.13) and (1.4.14) that

$$\left|\omega_j(t+t_n)-\omega_j(t)\right| \geq \int_{u_n}^{t} |f_j(s+t_n,\omega(s+t_n)) - f_j(s,\omega(s+t_n))|ds -$$

$$-\int_{u_n}^{t} |f_j(s,\omega(s+t_n)) - f_j(s,\omega(s))|ds \;-\int_{u_n}^{t} |\sum_{j=1}^{p} a_{ji}[\omega_j(s+t_n)-\omega_j(s)]|ds -$$

$$-|\omega_j(t_n+u_n)-\omega_j(u_n)| \geq \frac{\kappa}{2}\varepsilon_0 - \kappa L\varepsilon_0(\frac{1}{l}+\frac{2}{k}) - \kappa||A||\varepsilon_0(\frac{1}{l}+\frac{2}{k}) - \frac{\varepsilon_0}{l} \geq \frac{\varepsilon_0}{2l})$$

for $t \in [u_n + \kappa/2, u_n + \kappa]$.

Thus, the solution $\omega(t)$ is unpredictable. The theorem is proved.

Example 5. Consider the function $g(t,x) = (arctan(x) + 2)\Theta(t)$ of two vatiables t and x, where $\Theta(t) = \int_{-\infty}^{t} e^{-2(t-s)}\Omega(s)ds$, is the unpredictable function. It is easy to see that function $g(t,x)$ is continuously differentiable, and bounded such that $\sup_{\mathbb{R}\times G} |g(t,x)| = \frac{\pi}{4} + 1$. Moreover, $\sup_{\mathbb{R}\times G} |\frac{\partial g(t,x)}{\partial x}| = \frac{1}{2}$.

Let us fix arbitrary compact interval $I \subset \mathbb{R}$ and a positive number ε. We have that $|\Theta(t+t_n) - \Theta(t)| < \varepsilon$ for $t \in I$ and sufficiently large n. Consequently,

$$|g(t+t_n,x) - g(t,x)| \leq |arctan(x) + 2||\Theta(t+t_n) - \Theta(t)| < (\frac{\pi}{2} + 2)\varepsilon.$$

That is $g(t+t_n,x) \to g(t,x)$ as $n \to \infty$ uniformly for $(t,x) \in I \times G$.

On the other hand, it is true that $|\Theta(t+t_n) - \Theta(t)| \geq \bar{\varepsilon}_0$ for all $t \in [u_n - \kappa, u_n + \kappa]$ and $n \in \mathbb{N}$. This is why, we obtain that

$$|g(t+t_n,x) - g(t,x)| = |arctan(x) + 2||\Theta(t+\zeta_n) - \Theta(t)| \geq (-\frac{\pi}{2} + 2)\bar{\varepsilon}_0,$$

for $(t,x) \in [u_n - \kappa, u_n + \kappa] \times G, n \in \mathbb{N}$. Thus, $g(t,x)$ is an unpredictable (strongly) in t function.

Example 6. Let us consider the system of differential equations

$$\begin{cases} x_1' = -3x_1 - x_2 - x_3 + 0.51g(t,x_3) \\ x_2' = -x_1 - 3x_2 - x_3 + 0.62g(t,x_1) \\ x_3' = x_1 + x_2 - x_3 + 0.51g(t,x_2), \end{cases} \quad (1.4.17)$$

where $g(t,x)$ is the function from Example 5. The eigenvalues of the matrix of coefficients are equal to -2, -2 and -3. One can find that conditions $(C2) - (C4)$ are valid for system (1.4.17) with $\gamma = -2, K = 6$ and L=0.31. According to Theorem 1.4.1, there exists the unique asymptotically stable unpredictable solution of system (1.4.17). Figures 9, 10 show the simulation results for function $\psi(t)$, which exponentially converges to the solution of system (1.4.17), with initial data $\psi_1(0) = 0.05, \psi_2(0) = -0.1, \psi_3(0) = 0.15$. They confirm the irregularity of the dynamics.

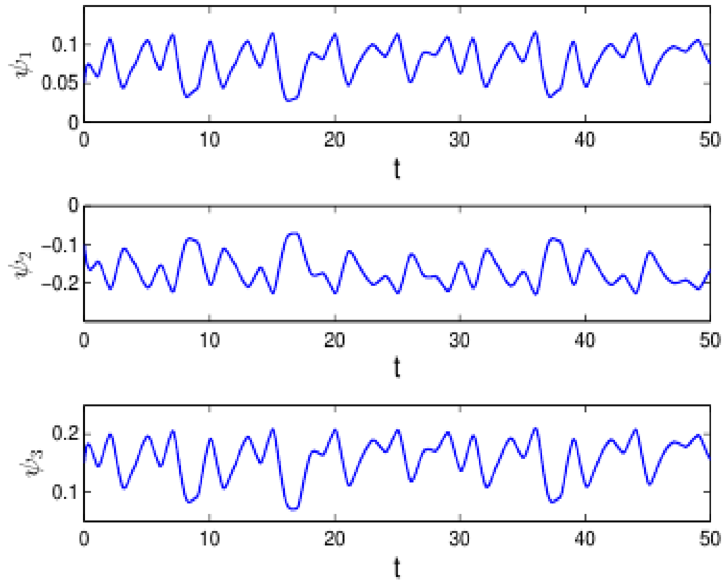

Figure 9 – The coordinates of the function $\psi(t)$

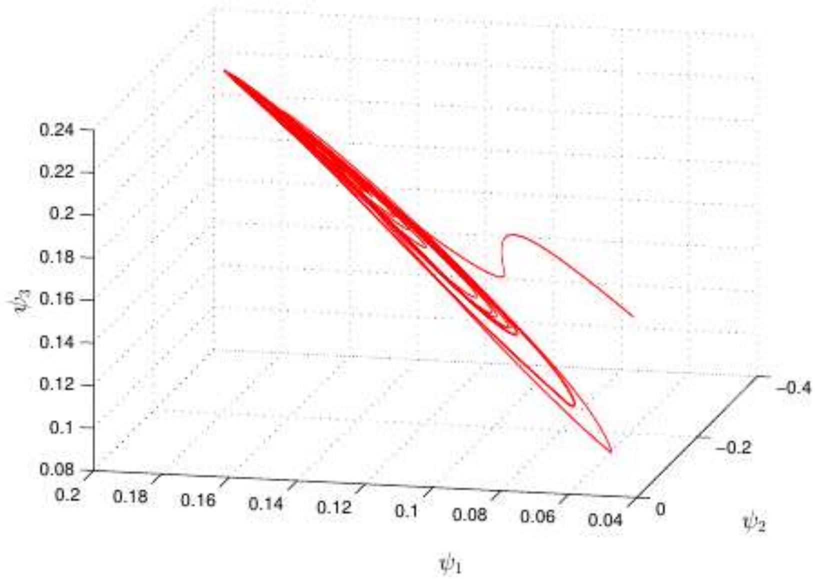

Figure 10 – The trajectory of the function $\psi(t)$

2 UNPREDICTABLE OSCILLATIONS IN NEURAL NETWORKS

2.1 SICNNs with unpredictable oscillations

SICNNs, which have been introduced by Bouzerdoum and Pinter [85, p. 215], plays an exceptional role in psychophysics, robotics, adaptive pattern recognition, vision and image processing. Therefore, they have been the subject of intense analysis be numerous authors in recent decades [89-94].

In its original formulation [85, p. 215], the SICNNs model is a two-dimensional grid of processing cells. Let C_{ij} denote the cell at the (i,j) position of the lattice. Denote by $N_r(ij)$ the r – neighborhood C_{ij}, such that

$$N_r(ij) = \{C_{kp}: \max(|k-i|, |p-j|) \leq r, 1 \leq k \leq m, 1 \leq p \leq n\},$$

where *m* and *n* are fixed natural numbers. In SICNNs, neighboring cells exert mutual inhibitory interactions of the shunting type. The dynamics of the cell C_{ij} is described by the following nonlinear ordinary differential equation,

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp} f(x_{kp}(t))x_{ij}(t) + v_{ij}(t), \qquad (2.1.1)$$

where x_{ij} is activity of the cell C_{ij}, the constant a_{ij} represents the passive decay rate of the cell activity, $C_{ij}^{kp} \geq 0$ is the connection or coupling strength of postsynaptic activity of the cell C_{kp} transmitted to the cell C_{ij} and the activation $f(x_{kp})$ is a positive continuous function representing the output or firing rate of the cell C_{kp}, $v_{ij}(t)$ is the external input to the cell C_{ij}.

Let us denote by $\mathcal{A}$ the set of functions $u(t) = (u_{11}, \dots, u_{1n}, \dots, u_{m1} \dots, u_{mn})$, $t, u_{ij} \in \mathbb{R}, i = 1,2,\dots,m, j = 1,2,\dots,n,$ where $m, n \in \mathbb{N}$, such that:

($\mathcal{A}1$) functions $u(t)$ are uniformly continuous and there exists a positive number H such that $||u||_1 < H$ for all $u(t) \in \mathcal{A}$;

($\mathcal{A}2$) there exists a sequence $t_p, t_p \to \infty$ as $p \to \infty$ such that for each $u(t) \in \mathcal{A}$ the sequence $u(t+t_p)$ uniformly converges to $u(t)$ on each closed and bounded interval of the real axis.

The following assumptions will be needed $(C1)$ the function $v(t) =$ $= (v_{11}, \dots, v_{1n}, \dots, v_{m1} \dots, v_{mn}),$ $t, v_{ij} \in \mathbb{R}, i = 1,2,\dots,m, j = 1,2,\dots,n,$ in system (2.1.1) belongs to $\mathcal{A}$ and is unpredictable such that there exist positive numbers $\delta, \varepsilon_0 > 0$ and a sequence $t_p \to \infty$ as $p \to \infty$, which satisfy $||v(t+t_p) - v(t)|| \geq \varepsilon_0$ for all $t \in [s_p - \delta, s_p + \delta]$, and $p \in \mathbb{N}$.

$(C2)$ for the rates we assume that $\gamma = \min_{(i,J)} a_{ij} > 0$ and $\bar{\gamma} = \max_{(i,j)} a_{ij}$;

$(C3)$ there exist positive numbers m_{ij} and m_f such that $\sup_{t\in\mathbb{R}} |v_{ij}| \leq m_{ij}$ for all $i = 1, \dots, m, j = 1, \dots, n,$ and $\sup_{|s|<H} |f(s)| \leq m_f$;

$(C4)$ there exists Lipschitz constant L such that $|f(s_1) - f(s_2)| \leq L|s_1 - s_2|$ for all $s_1, s_2, |s_1| < H, |s_2| < H$;

$(C5)$ $(LH + m_f) \max\limits_{(i,j)} \sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} < \gamma$ for all $i = 1, \dots, m, j = 1, \dots, n$.

Likewise, to the result in [88, p. 10], one can verify that the following assertion is valid.

Lemma 2.1.1. A bounded on $\mathbb{R}$ function $y(t) = \{y_{ij}(t)\}$, $i = 1, \dots, m$, $j = 1, \dots, n$, is a solution of SICNNs (2.1.1) if and only if the following integral equation is satisfied:

$$y_{ij}(t) = -\int_{-\infty}^{t} e^{-a_{ij}(t-s)} \left[\sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} f(y_{kp}(s)) y_{ij}(s) - v_{ij}(s) \right] ds \qquad (2.1.2)$$

for all $i = 1, \dots, m, j = 1, \dots, n$.

Define on $\mathcal{A}$ the operator Π such that $\Pi u(t) = \Pi_{ij} u(t), i = 1, \dots, m,$
$j = 1, \dots, n$, where:

$$\Pi_{ij} u(t) = -\int_{-\infty}^{t} e^{-a_{ij}(t-s)} \left[\sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} f(u_{kp}(s)) u_{ij}(s) - v_{ij}(s) \right] ds. \quad (2.1.3)$$

Lemma 2.1.2. If $u(t) \in \mathcal{A}$ then $\Pi u(t) \in \mathcal{A}$.

Proof. Fix a function $u(t) \in \mathcal{A}$, is not difficult to show that $\Pi u(t)$ satisfies the condition $(\mathcal{A}1)$.

Now, let us fix a positive number ε and a finite interval $[a, b] \subset \mathbb{R}$. Consider numbers $c < a$ and $\xi > 0$, which satisfy the following inequalities,

$$\frac{2}{\gamma} \left(\max_{(i,j)} \sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} (m_f H + LH^2) + 1 \right) e^{-\gamma(a-c)} < \frac{\varepsilon}{2}, \qquad (2.1.4)$$

and

$$\frac{\xi}{\gamma} \left(\max_{(i,j)} \sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} (m_f + LH) + 1 \right) < \frac{\varepsilon}{2}. \qquad (2.1.5)$$

We will show that $||\Pi u(t + t_p) - \Pi u(t)|| < \varepsilon$ on $[a, b]$ for sufficiently large p. Let p be sufficiently large number such that $||u(t + t_p) - u(t)|| < \xi$ and
$||v(t + t_p) - v(t)|| < \xi$ on $[c, b]$. Then for all $t \in [a, b]$ it is true that:

$$\left|\Pi_{ij}u(t+t_p)-\Pi_{ij}u(t)\right| \leq$$

$$\leq \int_{-\infty}^{c} e^{-\gamma(t-s)} \Big(\sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp} |f(u_{kp}(s))[u_{ij}(s)-u_{ij}(s+t_p)] +$$

$$+\left[f\left(u_{kp}(s)\right)-f\left(u_{kp}(s+t_p)\right)\right]u_{ij}(s+t_p)\Big|+\left|v_{ij}(s+t_p)-v_{ij}(s)\right|\Big)ds +$$

$$+\int_{c}^{t} e^{-\gamma(t-s)} \Big(\sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp} |f(u_{kp}(s))[u_{ij}(s)-u_{ij}(s+t_p)] +$$

$$+[f(u_{kp}(s))-f(u_{kp}(s+t_p))]u_{ij}(s+t_p)| + |v_{ij}(s+t_p)-v_{ij}(s)|)ds| +$$

$$\leq \frac{1}{\gamma}\left(\max_{(i,j)} \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp}\left(2m_f H + 2LH^2\right) + 2H\right)e^{-\gamma(a-c)} +$$
$$+\frac{\xi}{\gamma}\left(\max_{(i,j)} \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp}\left(m_f + LH\right) + 1\right),$$

for all $i = 1, \dots, m, j = 1, \dots, n$. Now inequalities (2.1.4) and (2.1.5) imply that $||\Pi u(t+t_p) - \Pi u(t)|| < \varepsilon$ for $t \in [a, b]$. Since ε is arbitrary small number, the condition ($\mathcal{A}2$) is valid. The lemma is proved.

Lemma 2.1.3. The operator Π is contractive in $\mathcal{A}$.

Proof. For two functions $\varphi, \psi \in \mathcal{A}$, and fixed $i = 1, \dots, m, j = 1, \dots, n$ we have that

$$\left|\Pi_{ij}\varphi(t) - \Pi_{ij}\psi(t)\right| \leq$$

$$\leq \int_{-\infty}^{t} e^{-\gamma(t-s)} \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp} \left|f\left(\varphi_{kp}(s)\right)\varphi_{ij}(s) - f\left(\varphi_{kp}(s)\right)\psi_{ij}(s)\right| ds +$$

$$+\int_{-\infty}^{t} e^{-\gamma(t-s)} \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp} \left|f(\varphi_{kp}(s))\psi_{ij}(s) - f(\psi_{kp}(s))\psi_{ij}(s)\right| ds \leq$$

$$\leq \frac{(LH+m_f)}{\gamma} \max_{(i,j)} \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp} ||\varphi-\psi||_1.$$

This is why $||\Pi\varphi - \Pi\psi||_1 \le \frac{(LH+m_f)}{\gamma} \max_{(i,j)} \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp} ||\varphi - \psi||_1$. Then condition $(C5)$ implies that the operator Π is contractive in the set $\mathcal{A}$. The lemma is proved.

Theorem 2.1.1. Suppose that conditions $(\mathrm{C}1) - (\mathrm{C}5)$ are valid, then the system (2.1.1) possesses a unique asymptotically stable unpredictable solution.

Proof. Let us show that the space $\mathcal{A}$ is complete. Consider a Cauchy sequence $\phi_k(t)$ in $\mathcal{A}$, which converges to a limit function $\phi(t)$ on $\mathbb{R}$. It suffices to show that $\phi(t)$ satisfies condition $(\mathcal{A}2)$, since condition $(\mathcal{A}1)$ can be easily checked. Fix a closed and bounded interval $I \subset \mathbb{R}$. We have that

$$\left|\left|\phi(t+t_p) - \phi(t)\right|\right| \le \left|\left|\phi(t+t_p) - \phi_k(t+t_p)\right|\right| + ||\phi_k(t+t_p) - \phi_k(t)|| +$$

$$+||\phi_k(t) - \phi(t)||. \qquad (2.1.6)$$

Now, one can take sufficiently large p and k such that each term on the right hand-side of (2.1.6) is smaller than $\frac{\varepsilon}{3}$ for an arbitrary positive ε and $t \in I$. The inequality implies that $||\phi(t+t_p) - \phi(t)|| \le \varepsilon$ on *I*. That is the sequence $\phi(t+t_p)$ uniformly converges to $\phi(t)$ on *I*. The completeness of $\mathcal{A}$ is proved. Now, by the contractive mapping theorem, duo to Lemmas 2.1.1 and 2.1.2, there exists a unique solution $\omega(t) \in \mathcal{A}$ of the equation (2.1.1).

One can find a positive number κ and natural numbers l and k such that the following inequalities are valid:

$$\kappa < \delta; \qquad (2.1.7)$$

$$\kappa\left(\frac{1}{2} - (\frac{1}{l} + \frac{2}{k})(\bar{\gamma} + \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp}\,(m_f + LH))\right) \ge \frac{3}{2l}; \qquad (2.1.8)$$

$$|\omega_{ij}(t+s) - \omega_{ij}(t)| < \varepsilon_0 min(\frac{1}{k}, \frac{1}{4l}), t \in \mathbb{R}, |s| < \kappa, \qquad (2.1.9)$$

for all $i = 1,2,\ldots,m, j = 1,2,\ldots,n$.

Denote $\Delta = |\omega_{ij}(t_p + s_p) - \omega_{ij}(s_p)|$ and consider two cases: (i) $\Delta < \varepsilon_0/l$; (ii) $\Delta \ge \varepsilon_0/l$ such that the remaining proof falls into two parts.

(i) From (2.1.9) it follows that:

$$|\omega_{ij}(t+s) - \omega_{ij}(t)| < \frac{\varepsilon_0}{l} + \frac{\varepsilon_0}{k} + \frac{\varepsilon_0}{k} = \varepsilon_0(\frac{1}{l} + \frac{2}{k}), \qquad (2.1.10)$$

if $t \in [s_p, s_p + \kappa]$. It is true that:

$$\omega_{ij}(t+t_p)-\omega_{ij}(t)=\omega(t_p+s_p)-\omega(s_p)-$$

$$-\int_{s_p}^{t}a_{ij}\left(\omega(s+t_p)-\omega(s)\right)ds-\int_{s_p}^{t}(v_{ij}(s+t_p)-v_{ij}(s))ds-$$

$$-\int_{s_p}^{t}\sum_{C_{kp}\in N_r(i,j)}C_{ij}^{kp}\,(f\,(\omega_{kp}(s+t_p))\omega_{ij}(s+t_p)-f(\omega_{kp}(s))\omega_{ij}(s))ds. \quad (2.1.11)$$

We obtain from (2.1.7), (2.1.8) and (2.1.10) - (2.1.11) that:

$$|\omega_{ij}(t+s)-\omega_{ij}(t)|\geq\int_{s_p}^{t}|v_{ij}(s+t_p)-v_{ij}(s)|ds-|\omega(t_p+s_p)-\omega(s_p)|-$$

$$-\int_{s_p}^{t}a_{ij}|\omega(s+t_p)-\omega(s)|ds-\int_{s_p}^{t}\sum_{C_{kp}\in N_r(i,j)}C_{ij}^{kp}\,|f\,(\,\omega_{kp}(s+t_p))\omega_{ij}(s+t_p)-$$

$$-f\left(\omega_{kp}(s)\right)\omega_{ij}(s)|ds\geq\varepsilon_0\frac{\kappa}{2}-\frac{\varepsilon_0}{l}-\varepsilon_0\kappa\left(\frac{1}{l}+\frac{2}{k}\right)(\bar{\gamma}+\sum_{C_{kp}\in N_r(i,j)}C_{ij}^{kp}\,(m_f+LH))$$

$$=\ \varepsilon_0\kappa\left(\frac{1}{2}-(\frac{1}{l}+\frac{2}{k})(\bar{\gamma}+\sum_{C_{kp}\in N_r(i,j)}C_{ij}^{kp}\,(m_f+LH))\right)\geq\frac{3\varepsilon_0}{2l},$$

for $t\in\left[s_p+\frac{\kappa}{2},s_p+\kappa\right]$.

(ii) For the case $\Delta\geq\varepsilon_0/l$, it can easily find that (2.1.9) implies:

$$\left\|\omega(t+t_p)-\omega(t)\right\|\geq\left\|\omega(t_p+s_p)-\omega(s_p)\right\|-\left\|\omega(s_p)-\omega(t)\right\|-$$

$$-||\omega(t+t_p)-\omega(t_p+s_p)||\ \geq\ \frac{\varepsilon_0}{l}-\frac{\varepsilon_0}{4l}-\frac{\varepsilon_0}{4l}=\frac{\varepsilon_0}{2l},$$

if $t\in\left[s_p-\kappa,s_p+\kappa\right]$ and $p\in\mathbb{N}$.

Thus, one can conclude that $\omega(t)$ is unpredictable.

Finally, we will discuss the stability of the unpredictable solution $\omega(t)$. It is true that:

$$\omega_{ij}(t)=e^{-a_{ij}(t-t_0)}\omega_{ij}(t_0)-$$

$$-\int_{t_0}^{t}e^{-a_{ij}(t-s)}\left[\sum_{C_{kp}\in N_r(i,j)}C_{ij}^{kp}f(\omega_{kp}(s))\omega_{ij}(s)-v_{ij}(s)\right]ds,$$

for all $i = 1,2,\dots,m, j = 1,2,\dots,n$.

Let $z(t) = (z_{11}(t), z_{12}(t), \dots, z_{mn}(t))$ be another solution of system (2.1.1). One can write:

$$z_{ij}(t) = e^{-a_{ij}(t-t_0)} z_{ij}(t_0) -$$

$$-\int_{t_0}^{t} e^{-a_{ij}(t-s)} \left[\sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} f(z_{kp}(s)) z_{ij}(s) - v_{ij}(s) \right] ds,$$

for all $i = 1,2,\dots,m, j = 1,2,\dots,n$. Making use of the relation:

$$z_{ij}(t) - \omega_{ij}(t) = e^{-a_{ij}(t-t_0)} (z_{ij}(t_0) - \omega_{ij}(t_0)) -$$

$$-\int_{t_0}^{t} e^{-a_{ij}(t-s)} \left[\sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} f(\omega_{kp}(s)) \omega_{ij}(s) \sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} f(z_{kp}(s)) z_{ij}(s) \right] ds,$$

we obtain that:

$$|z_{ij}(t) - \omega_{ij}(t)| \le e^{-\gamma(t-t_0)} |z_{ij}(t_0) - \omega_{ij}(t_0)| +$$

$$+m_f \int_{t_0}^{t} e^{-\gamma(t-s)} \sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} \left| z_{ij}(s) - \omega_{ij}(s) \right| ds +$$

$$+LH \int_{t_0}^{t} e^{-\gamma(t-s)} \sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} \left| z_{kp}(s) - \omega_{ij}(s) \right| ds \le$$

$$\le e^{-\gamma(t-t_0)} ||z_{ij}(t_0) - \omega_{ij}(t_0)|| +$$

$$+m_f \int_{t_0}^{t} e^{-\gamma(t-s)} \sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} \left\| z_{ij}(s) - \omega_{ij}(s) \right\| ds +$$

$$+LH \int_{t_0}^{t} e^{-\gamma(t-s)} \sum_{C_{kp} \in N_r(i,j)} C_{ij}^{kp} ||z_{kp}(s) - \omega_{ij}(s)|| ds,$$

for all $i = 1,2,\dots,m, j = 1,2,\dots,n$.
Thus, one can be confirmed that:

$$||z(t)-\omega(t)|| \leq e^{-\gamma(t-t_0)}||z(t_0)-\omega(t_0)|| + D\int_{t_0}^{t} e^{-\gamma(t-s)}||z(s)-\omega(s)||ds,$$

where $D = (LH + m_f)\max_{(i,j)} \sum_{C_{kp}\in N_r(i,j)} C_{ij}^{kp}$. Multiplying both sides of the last inequality by $e^{\gamma t}$ we obtain that:

$$e^{\gamma t}||z(t)-\omega(t)|| \leq e^{\gamma t_0}||z(t_0)-\omega(t_0)|| + D\int_{t_0}^{t} e^{\gamma s}||z(s)-\omega(s)||ds.$$

Now, applying Gronwall-Belman Lemma, one can attain that:

$$||z(t)-\omega(t)|| \leq ||z(t_0)-\omega(t_0)||e^{(D-\gamma)(t-t_0)}.$$

The last inequality and condition $(C5)$ confirm that the unpredictable solution $\omega(t)$ is uniformly asymptotically stable. The theorem is proved.

Example 7. Let us introduce the following SICNNs:

$$\frac{dx_{ij}}{dt} = -a_{ij}x_{ij} - \sum_{C_{kp}\in N_1(i,j)} C_{ij}^{kp} f(x_{kp}(t))x_{ij}(t) + v_{ij}(t), \qquad (2.1.12)$$

where $= 1,2,3,$ $a_{11} = 4, a_{12} = 2, a_{13} = 7, a_{21} = 5, a_{22} = 9, a_{23} = 6, C_{11} = 0.03,$ $C_{12} = 0.05, C_{13} = 0.01, C_{21} = 0.06, C_{22} = 0.04, C_{23} = 0.05.$ The functions are defined as $f(s) = 0.25arctg(s),$ $v_{11}(t) = 4\Theta^3(t) + 5,$ $v_{12}(t) = -5\Theta(t) + 1,$ $v_{13}(t) = 2\Theta^3(t),$ $v_{21}(t) = 6\Theta(t) - 1,$ $v_{22}(t) = 3\Theta(t) + 2,$ $v_{23}(t) = -\Theta(t) + 3,$ where $\Theta(t) = \int_{-\infty}^{t} e^{-2.5(t-s)}\Omega(s)ds$ is the unpredictable function. Moreover, according to the Definition 1.1.1 and properties of unpredictable functions, the function $v(t)$ is unpredictable. We have that $|v_{ij}(t)| \leq m_{ij}$, where $m_{11} = 1.26;$ $m_{12} = 3;$ $m_{13} = 0.13;$ $m_{21} = 3.4;$ $m_{22} = 3.2;$ $m_{23} = 3.4.$ One can calculate that $\sum_{C_{kp}\in N_1(1,1)} C_{11}^{kp} = 0.18,$ $\sum_{C_{kp}\in N_1(1,2)} C_{12}^{kp} = 0.24,$ $\sum_{C_{kp}\in N_1(1,3)} C_{13}^{kp} = 0.15,$ $\sum_{C_{kp}\in N_1(2,1)} C_{21}^{kp} = 0.18,$ $\sum_{C_{kp}\in N_1(2,2)} C_{22}^{kp} = 0.24,$ $\sum_{C_{kp}\in N_1(2,3)} C_{23}^{kp} = 0.15.$ The conditions $(C1)-(C5)$ hold for the network (2.1.12) with $\gamma = 2, \bar{\gamma} = 9,$ $L = 0.25,$ $m_f = 0.3925$, and $H = 1.5$.

Figure 11 shows the coordinates of the function $\omega(t)$ with initial values $\omega_{11}(0) = 1.21,$ $\omega_{12}(0) = 0,$ $\omega_{13}(0) = 0.02,$ $\omega_{21}(0) = 0.25,$ $\omega_{22}(0) = 0.23,$ $\omega_{23}(0) = 0.41$. The function $\omega(t)$ exponentially tends to the unpredictable solution $x(t)$ of the equation (2.1.12), as time increases.

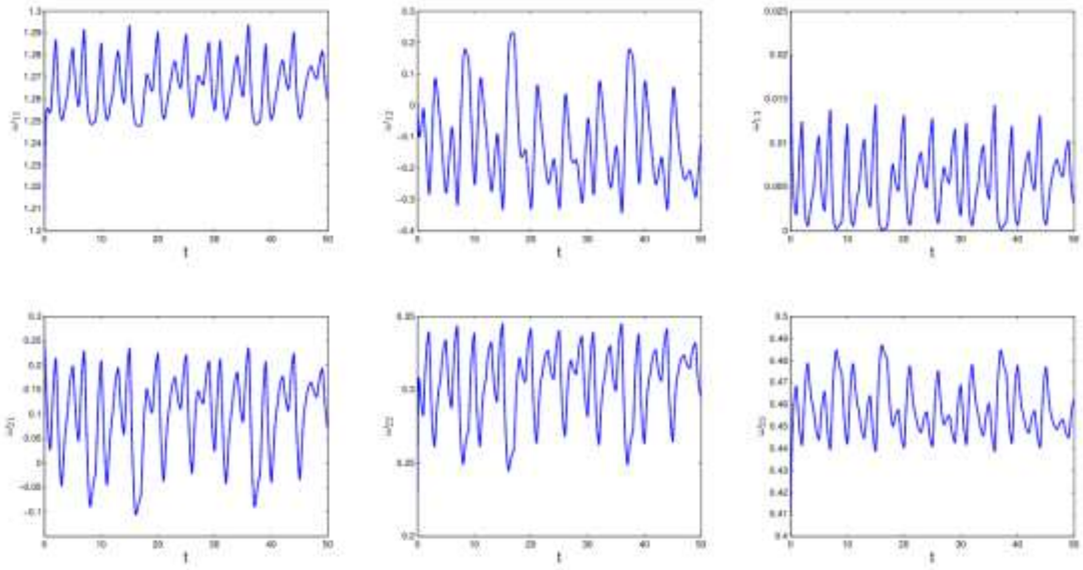

Figure 11 – The coordinates of the function $\omega(t)$, which exponentially tends to the unpredictable solution $x(t)$ of the equation (2.1.12), as time increases

2.2 Strongly unpredictable oscillations of SICNNs

Despite the fact that the concepts of unpredictable and strongly unpredictable solutions are close, and the proof from existence and uniqueness is largely similar, we believe that it is logical to reproduce the complete proof of a strongly unpredictable solution in our research. In addition, the dynamics of each coordinate is very important in neural networks and this entails additional conditions.

Next, we consider following SICNNs:

$$\frac{dx_{ij}}{dt} = -b_{ij}x_{ij} - \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp} f(x_{kp}(t))x_{ij}(t) + g_{ij}(t), \qquad (2.2.1)$$

with strongly unpredictable perturbations.

Let us denote by $\mathcal{B}$ the set of functions $u(t) = (u_{11}, \dots, u_{1n}, \dots, u_{m1} \dots, u_{mn}), t,$ $u_{ij} \in \mathbb{R}, i = 1,2,\dots,m, j = 1,2,\dots,n,$ where $m, n \in \mathbb{N}$, such that:

($\mathcal{B}$1) functions $u(t)$ are uniformly continuous;

($\mathcal{B}$2) there exists a positive number H such that $||u||_1 < H$ for all $u(t) \in \mathcal{B}$;

($\mathcal{B}$3) there exists a sequence $t_p, t_p \to \infty$ as $p \to \infty$ such that for each $u(t) \in \mathcal{B}$ the sequence $u(t + t_p)$ uniformly converges to $u(t)$ on each closed and bounded interval of the real axis.

The following conditions will be needed:

$(D1)$ the function $g(t) = (g_{11}, \dots, g_{1n}, \dots, g_{m1} \dots, g_{mn}), t, g_{ij} \in \mathbb{R}, i = 1,2, \dots, m,\ j = 1,2, \dots, n,$ in system (2.2.1) belongs to $\mathcal{B}$ and is strongly unpredictable such that there exist positive numbers $\delta, \varepsilon_0 > 0$ and a sequence $t_p \to \infty$ as $p \to \infty$, which satisfy $|g_{ij}(t + t_p) - g_{ij}(t)| \geq \varepsilon_0$ for all $t \in [s_p - \delta, s_p + \delta], i = 1, \dots, m,$ $j = 1,\dots,n,$ and $p \in \mathbb{N}$.

$(D2)$ $\gamma \leq b_{ij} \leq \bar{\gamma}$, where $\gamma, \bar{\gamma}$ are positive numbers;

$(D3)$ $|g_{ij}(t)| \leq m_{ij}$, where m_{ij} are positive numbers, for all $i = 1, \dots, m$, $j = 1, \dots, n$, and $t \in \mathbb{R}$;

$(D4)$ $|f(s)| \leq m_f$, for $|s| < H$ and some constant $m_f > 0$;

$(D5)$ there exists a positive constant L such that $|f(s_1) - f(s_2)| \leq L|s_1 - s_2|$ for all $s_1, s_2, |s_1| < H, |s_2| < H$;

$(D6)$ $m_f \sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} < b_{ij}$, for each $i = 1, \dots, m, j = 1, \dots, n$;

$(D7)$ $\dfrac{m_{ij}}{b_{ij} - m_f \sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp}} < H$, for all $i = 1, \dots, m, j = 1, \dots, n$;

$(D8)$ $(LH + m_f) \max_{(i,j)} \sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} < \gamma$.

Likewise, to the result in Hartman [88, p. 10], one can verify that the following assertion is valid.

Lemma 2.2.1. A bounded on $\mathbb{R}$ function $y(t) = \{y_{ij}(t)\}$, $i = 1, \dots, m$, $j = 1, \dots, n$, is a solution of SICNNs (2.2.1) if and only if the following integral equation is satisfied

$$y_{ij}(t) = -\int_{-\infty}^{t} e^{-b_{ij}(t-s)} \left[\sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} f(y_{kp}(s)) y_{ij}(s) - g_{ij}(s) \right] ds \qquad (2.2.2)$$

for all $i = 1, \dots, m, j = 1, \dots, n$.

Define on $\mathcal{B}$ the operator Π such that $\Pi u(t) = \Pi_{ij} u(t), i = 1, \dots, m$, $j = 1, \dots, n$, where

$$\Pi_{ij} u(t) = -\int_{-\infty}^{t} e^{-b_{ij}(t-s)} \left[\sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} f(u_{kp}(s)) u_{ij}(s) - g_{ij}(s) \right] ds. \qquad (2.2.3)$$

Lemma 2.2.2. If $u(t) \in \mathcal{B}$ then $\Pi u(t) \in \mathcal{B}$.

Proof. Fix a function $u(t) \in \mathcal{B}$. One can find that

$$|\Pi_{ij} u(t)| \leq \int_{-\infty}^{t} e^{-b_{ij}(t-s)} \left(\sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} |f(u_{kp}(s)) u_{ij}(s)| + |g_{ij}(s)| \right) ds \leq$$

$$\leq \frac{1}{b_{ij}} \left(H m_f \sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} + m_{ij} \right) \qquad (2.2.4)$$

for all $i = 1, \dots, m, j = 1, \dots, n$. According to conditions $(D6)$, $(D7)$ we have that $||\Pi u(t)||_1 < H$.

Uniform continuity of function $\Pi u(t)$ follows from estimates:

$$\left|\frac{\partial \Pi_{ij} u(t)}{\partial t}\right| = |\sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} f\left(u_{kp}(s)\right) u_{ij}(s) - g_{ij}(s) -$$

$$-b_{ij} \int_{-\infty}^{t} e^{-b_{ij}(t-s)} \left[\sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} f\left(u_{kp}(s)\right) u_{ij}(s) - g_{ij}(s)\right] ds| \leq$$

$$\leq (H m_f \sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} + H) + \frac{\bar{\gamma}}{\gamma}(H m_f \sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} + H) =$$

$$= (1 + \frac{\bar{\gamma}}{\gamma}) H (m_f \sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} + 1) < \infty$$

for all $i = 1, \dots, m, j = 1, \dots, n$. That is the condition $(\mathcal{B}1)$ is valid, since the derivatives are bounded.

Let us fix a positive number ε and a finite interval $[a, b] \subset \mathbb{R}$. Consider numbers $c < a$ and $\xi > 0$, which satisfy the following inequalities,

$$\frac{2H}{\gamma}\left(\max_{(i,j)} \sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} (m_f + LH) + 1\right) e^{-\gamma(a-c)} < \frac{\varepsilon}{2}, \tag{2.2.5}$$

$$\frac{\xi}{\gamma}\left(\max_{(i,j)} \sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} (m_f + LH) + 1\right) < \frac{\varepsilon}{2}. \tag{2.2.6}$$

We will show that $||\Pi u(t + t_p) - \Pi u(t)|| < \varepsilon$ on $[a, b]$ for sufficiently large p. Let p be a large enough number such that $||u(t + t_p) - u(t)|| < \xi$ and $||g(\mathrm{t} + t_p) - g(\mathrm{t})|| < \xi$ on $[c, b]$. Then for all $t \in [a, b]$ it is true that:

$$\left|\Pi_{ij} u\left(t + t_p\right) - \Pi_{ij} u(t)\right| \leq$$

$$\leq \int_{-\infty}^{c} e^{-\gamma(t-s)} (\sum_{D_{kp} \in N_r(i,j)} D_{ij}^{kp} |f(u_{kp}(s))[u_{ij}(s) - u_{ij}(s + t_p)] +$$

$$+\left[f\left(u_{kp}(s)\right)-f\left(u_{kp}(s+t_p)\right)\right]u_{ij}(s+t_p)\big|+\big|g_{ij}(s+t_p)-g_{ij}(s)\big|\big)ds+$$

$$+\int_c^t e^{-\gamma(t-s)}\,(\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\,|f(u_{kp}(s))[u_{ij}(s)-u_{ij}(s+t_p)]+$$

$$+[f(u_{kp}(s))-f(u_{kp}(s+t_p))]u_{ij}(s+t_p)|+|g_{ij}(s+t_p)-g_{ij}(s)|)ds|+$$

$$\leq \frac{1}{\gamma}\left(\max_{(i,j)}\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\left(2m_f H+2LH^2\right)+2H\right)e^{-\gamma(a-c)}+$$

$$+\frac{\xi}{\gamma}\left(\max_{(i,j)}\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\,(m_f+LH)+1\right),$$

for all $i=1,\dots,m, j=1,\dots,n$. Now inequalities (2.2.5) and (2.2.6) imply that $||\Pi u(t+t_p)-\Pi u(t)||<\varepsilon$ for $t\in[a,b]$. Since ε is arbitrary small number, the condition ($\mathcal{B}$3) is valid. The lemma is proved.

Lemma 2.2.3. The operator Π is contractive in $\mathcal{B}$.

Proof. For two functions $\varphi,\psi\in\mathcal{B}$, we have that:

$$\left|\Pi_{ij}\varphi(t)-\Pi_{ij}\psi(t)\right|\leq$$

$$\leq\int_{-\infty}^t e^{-\gamma(t-s)}\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\left|f\left(\varphi_{kp}(s)\right)\varphi_{ij}(s)-f\left(\varphi_{kp}(s)\right)\psi_{ij}(s)\right|ds+$$

$$+\int_{-\infty}^t e^{-\gamma(t-s)}\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\,|f(\varphi_{kp}(s))\psi_{ij}(s)-f(\psi_{kp}(s))\psi_{ij}(s)|ds\leq$$

$$\leq\frac{(LH+m_f)}{\gamma}\max_{(i,j)}\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}||\varphi-\psi||_1.$$

This is why $||\Pi\varphi-\Pi\psi||_1\leq\frac{(LH+m_f)}{\gamma}\max_{(i,j)}\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}||\varphi-\psi||_1$. Then condition ($D$8) implies that the operator Π is contractive in the set $\mathcal{B}$. The lemma is proved.

Theorem 2.2.1. Assume that conditions (D1)-(D8) are fulfilled, then the system (2.2.1) possesses a unique asymptotically stable strongly unpredictable solution.

Proof. Let us show that the space $\mathcal{B}$ is complete. Consider a Cauchy sequence $\phi_k(t)$ in $\mathcal{B}$, which converges to a limit function $\phi(t)$ on $\mathbb{R}$. It suffices to show that $\phi(t)$ satisfies condition ($\mathcal{B}$3), since other two conditions can be easily checked. Fix a closed and bounded interval $I \subset \mathbb{R}$. We have that:

$$\left|\left|\phi\left(t+t_p\right)-\phi(t)\right|\right| \leq \left|\left|\phi\left(t+t_p\right)-\phi_k\left(t+t_p\right)\right|\right| + ||\phi_k\left(t+t_p\right)-\phi_k(t)|| +$$

$$+||\phi_k(t)-\phi(t)|| \qquad (2.2.7)$$

Now, one can take sufficiently large p and k such that each term on right hand-side of (2.2.7) is smaller than $\frac{\varepsilon}{3}$ for an arbitrary positive ε and $t \in I$. The inequality implies that $||\phi\left(t+t_p\right)-\phi(t)|| \leq \varepsilon$ on I. That is the sequence $\phi\left(t+t_p\right)$ uniformly converges to $\phi(t)$ on I. The completeness of $\mathcal{B}$ is proved. Now, by the contractive mapping theorem, duo to Lemmas 2.2.2 and 2.2.3, there exists a unique solution $\omega(t) \in \mathcal{B}$ of the equation (2.2.1).

Next, using the relations:

$$\omega_{ij}(t) = \omega_{ij}\left(s_p\right) - \int_{s_p}^{t} b_{ij}\omega_{ij}(s)ds -$$

$$-\int_{s_p}^{t} \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp} f\left(\omega_{kp}(s)\right)\omega_{ij}(s)ds + \int_{s_p}^{t} g_{ij}(s)ds$$

and

$$\omega_{ij}\left(t+t_p\right) = \omega_{ij}\left(t_p+s_p\right) - \int_{s_p}^{t} b_{ij}\omega_{ij}\left(s+t_p\right)ds -$$

$$-\int_{s_p}^{t} \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp} f\left(\omega_{kp}(s+t_p)\right)\omega_{ij}(s+t_p)ds + \int_{s_p}^{t} g_{ij}(s+t_p)ds,$$

we obtain that:

$$\omega_{ij}\left(t+t_p\right) - \omega_{ij}(t) = \omega\left(t_p+s_p\right) - \omega\left(s_p\right) -$$

$$-\int_{s_p}^{t} b_{ij}\left(\omega\left(s+t_p\right)-\omega(s)\right)ds - \int_{s_p}^{t}\left(g_{ij}(s+t_p)-g_{ij}(s)\right)ds -$$

$$-\int_{s_p}^{t} \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\left(f\left(\omega_{kp}(s+t_p)\right)\omega_{ij}(s+t_p) - f\left(\omega_{kp}(s)\right)\omega_{ij}(s)\right)ds. \qquad (2.2.8)$$

There exist a positive number κ and integers l, k such that the following inequalities are valid:

$$\kappa < \delta; \tag{2.2.9}$$

$$\kappa\left(\frac{1}{2} - (\frac{1}{l} + \frac{2}{k})(\bar{\gamma} + \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\,(m_f + LH))\right) \geq \frac{3}{2l}; \tag{2.2.10}$$

$$|\omega_{ij}(t+s) - \omega_{ij}(t)| < \varepsilon_0 min(\frac{1}{k}, \frac{1}{4l}), t \in \mathbb{R}, |s| < \kappa, \tag{2.2.11}$$

for all $i = 1,2,\ldots,m, j = 1,2,\ldots,n$.

Let the numbers κ, l and k as well as numbers $p \in \mathbb{N}$, and $i = 1,2,\ldots,m, j = 1,2,\ldots,n$, be fixed. Denote $\Delta = |\omega_{ij}(t_p + s_p) - \omega_{ij}(s_p)|$ and consider two alternative cases: (i) $\Delta < \varepsilon_0/l$; (ii) $\Delta \geq \varepsilon_0/l$ such that the remaining proof falls naturally into two parts.

(i) From (2.2.11) it follows that:

$$|\omega_{ij}(t+s) - \omega_{ij}(t)| < \frac{\varepsilon_0}{l} + \frac{\varepsilon_0}{k} + \frac{\varepsilon_0}{k} = \varepsilon_0(\frac{1}{l} + \frac{2}{k}), \tag{2.2.12}$$

if $t \in [s_p, s_p + \kappa]$.

The inequalities (2.2.9) -(2.2.12) imply that:

$$|\omega_{ij}(t+s) - \omega_{ij}(t)| \geq \int_{s_p}^{t} |g_{ij}(s+t_p) - g_{ij}(s)| ds - |\omega(t_p + s_p) - \omega(s_p)| -$$

$$- \int_{s_p}^{t} b_{ij} |\omega(s+t_p) - \omega(s)| ds - \int_{s_p}^{t} \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\,|f\,(\omega_{kp}(s+t_p))\omega_{ij}(s+t_p) -$$

$$-f\left(\omega_{kp}(s)\right)\omega_{ij}(s)| ds \geq \varepsilon_0 \frac{\kappa}{2} - \frac{\varepsilon_0}{l} - \varepsilon_0\kappa\left(\frac{1}{l} + \frac{2}{k}\right)(\bar{\gamma} + \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\,(m_f + LH))$$

$$= \varepsilon_0\kappa\left(\frac{1}{2} - (\frac{1}{l} + \frac{2}{k})(\bar{\gamma} + \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\,(m_f + LH))\right) \geq \frac{3\varepsilon_0}{2l},$$

for $t \in \left[s_p + \frac{\kappa}{2}, s_p + \kappa\right]$.

(ii) For the case $\Delta \geq \varepsilon_0/l$, and each $i = 1,2,\ldots,m, j = 1,2,\ldots,n$, it can easily find that (2.2.11) implies:

$$|\omega_{ij}(t+t_p)-\omega_{ij}(t)| \geq |\omega_{ij}(t_p+s_p)-\omega_{ij}(s_p)|-|\omega_{ij}(s_p)-\omega_{ij}(t)|-$$

$$-|\omega_{ij}(t+t_p)-\omega_{ij}(t_p+s_p)| \ \geq \ \frac{\varepsilon_0}{l}-\frac{\varepsilon_0}{4l}-\frac{\varepsilon_0}{4l}=\frac{\varepsilon_0}{2l},$$

if $t \in [s_p-\kappa, s_p+\kappa]$ and $p \in \mathbb{N}$. Thus, one can conclude that $\omega(t)$ is a strongly unpredictable with $\bar{s}_p = s_p + \frac{3\kappa}{4}, \bar{\delta} = \frac{\kappa}{4}$.

Finally, we will discuss the stability of the solution $\omega(t)$. It is true that:

$$\omega_{ij}(t) = e^{-b_{ij}(t-t_0)}\omega_{ij}(t_0) -$$

$$-\int_{t_0}^{t} e^{-b_{ij}(t-s)}\left[\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp} f(\omega_{kp}(s))\omega_{ij}(s)-g_{ij}(s)\right]ds,$$

$i \ = \ 1,2,\ldots,m, j \ = \ 1,2,\ldots,n.$

Let $z(t) = (z_{11}(t), z_{12}(t),\ldots,z_{mn}(t))$ be another solution of system (2.2.1). One can write:

$$z_{ij}(t) = e^{-b_{ij}(t-t_0)}z_{ij}(t_0) -$$

$$-\int_{t_0}^{t} e^{-b_{ij}(t-s)}\left[\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp} f(z_{kp}(s))z_{ij}(s)-g_{ij}(s)\right]ds,$$

for all $i \ = \ 1,2,\ldots,m, j \ = \ 1,2,\ldots,n$. Making use of the relation

$$z_{ij}(t)-\omega_{ij}(t) = e^{-b_{ij}(t-t_0)}(z_{ij}(t_0)-\omega_{ij}(t_0)) -$$

$$-\int_{t_0}^{t} e^{-b_{ij}(t-s)}\left[\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp} f(\omega_{kp}(s))\omega_{ij}(s)-\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp} f(z_{kp}(s))z_{ij}(s)\right]ds,$$

we obtain that

$$|z_{ij}(t)-\omega_{ij}(t)| \leq e^{-\gamma(t-t_0)}|z_{ij}(t_0)-\omega_{ij}(t_0)| +$$

$$+m_f\int_{t_0}^{t} e^{-\gamma(t-s)} \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\left|z_{ij}(s)-\omega_{ij}(s)\right|ds +$$

$$+LH\int_{t_0}^{t} e^{-\gamma(t-s)} \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\left|z_{kp}(s)-\omega_{ij}(s)\right|ds \leq$$

$$\leq e^{-\gamma(t-t_0)}||z_{ij}(t_0)-\omega_{ij}(t_0)|| +$$

$$+m_f\int_{t_0}^{t} e^{-\gamma(t-s)} \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}\left|\left|z_{ij}(s)-\omega_{ij}(s)\right|\right| ds +$$

$$+LH\int_{t_0}^{t} e^{-\gamma(t-s)} \sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}||z_{kp}(s)-\omega_{ij}(s)||ds,$$

for all $i = 1,2,\dots,m, j = 1,2,\dots,n.$
Thus, one can be confirmed that:

$$||z(t)-\omega(t)|| \leq e^{-\gamma(t-t_0)}||z(t_0)-\omega(t_0)|| + F\int_{t_0}^{t} e^{-\gamma(t-s)}||z(s)-\omega(s)||ds,$$

where $F = (LH + m_f)\max\limits_{(i,j)}\sum_{D_{kp}\in N_r(i,j)} D_{ij}^{kp}$. Multiplying both sides of the last inequality by $e^{\gamma t}$ we obtain that:

$$e^{\gamma t}||z(t)-\omega(t)|| \leq e^{\gamma t_0}||z(t_0)-\omega(t_0)|| + F\int_{t_0}^{t} e^{\gamma s}||z(s)-\omega(s)||ds.$$

Now, applying Gronwall-Belman Lemma, one can attain that:

$$||z(t)-\omega(t)|| \leq ||z(t_0)-\omega(t_0)||e^{(F-\gamma)(t-t_0)}.$$

The last inequality and condition $(\mathcal{B}8)$ confirm that the strongly unpredictable solution $\omega(t)$ is uniformly asymptotically stable. The theorem is proved.

Example 9. Consider the following SICNNs:

$$\frac{dx_{ij}}{dt} = -b_{ij}x_{ij} - \sum_{D_{kp}\in N_1(i,j)} D_{ij}^{kp}f(x_{kp}(t))x_{ij}(t) + g_{ij}(t), \qquad (2.2.13)$$

where $i,j = 1,2,3,$

$$\begin{pmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix} = \begin{pmatrix} 3 & 8 & 4\\ 5 & 9 & 6\\ 2 & 7 & 3\end{pmatrix}, \quad \begin{pmatrix} D_{11} & D_{12} & D_{13}\\ D_{21} & D_{22} & D_{23}\\ D_{31} & D_{32} & D_{33}\end{pmatrix} = \begin{pmatrix} 0.04 & 0.03 & 0\\ 0.07 & 0.06 & 0.02\\ 0.05 & 0 & 0.08\end{pmatrix},$$

and $f(s) = 0.05arctg(s)$, $g_{11}(t) = 25\Theta^3(t) + 1$, $g_{12}(t) = 4\Theta(t)$, $g_{13}(t) =$ $= -3\Theta(t) + 2$, $g_{21}(t) = 2\Theta(t) + 2$, $g_{22}(t) = 17\Theta^3(t)$, $g_{23}(t) = 19\Theta(t) - 1$, $g_{31}(t) = -7\Theta(t) + 2$, $g_{32}(t) = 3\Theta(t)$, $g_{33}(t) = -13\Theta^3(t) + 2$, where $\Theta(t) =$ $= \int_{-\infty}^{t} e^{-3(t-s)}\Omega(s)ds$ is the unpredictable function. Moreover, according to the properties of unpredictable functions, the functions $g_{ij}(t), i = 1,2,3, j = 1,2,3,$ are unpredictable. We have that $|g_{ij}(t)| \le m_{ij}$, where $m_{11} = 1.93$; $m_{12} = 1.34$; $m_{13} = 3$; $m_{21} = 8$; $m_{22} = 2.67$; $m_{23} = 7.34$; $m_{31} = 4.34$; $m_{32} = 1$; $m_{33} = 2.49$. One can calculate that $\sum_{D_{kp}\in N_1(1,1)} D_{11}^{kp} = 0.20$, $\sum_{D_{kp}\in N_1(1,2)} D_{12}^{kp} = 0.22$, $\sum_{D_{kp}\in N_1(1,3)} D_{13}^{kp} = 0.11$, $\sum_{D_{kp}\in N_1(2,1)} D_{21}^{kp} = 0.25$, $\sum_{D_{kp}\in N_1(2,2)} D_{22}^{kp} = 0.35$, $\sum_{D_{kp}\in N_1(2,3)} D_{23}^{kp} = 0.19$, $\sum_{D_{kp}\in N_1(3,1)} D_{31}^{kp} = 0.18$, $\sum_{D_{kp}\in N_1(3,2)} D_{32}^{kp} = 0.28$, $\sum_{D_{kp}\in N_1(3,3)} D_{33}^{kp} = 0.16$. The conditions ($\mathcal{B}1$) -($\mathcal{B}8$) hold for the network (2.2.13) with $\gamma = 2$, $\bar{\gamma} = 9$, $m_f = 0.079$, $L = 0.05$ and $H = 2$.

In figure 12 we depict the coordinates of the function $\phi(t)$ with initial values $\phi_{11}(0) = 1.0245$, $\phi_{12}(0) = 0.2996$, $\phi_{13}(0) = 0.0837$, $\phi_{21}(0) = 0.8283$, $\phi_{22}(0) = 0.0413$, $\phi_{23}(0) = 1.8122$, $\phi_{31}(0) = 1.0678$, $\phi_{32}(0) = 0.2013$, $\phi_{33}(0) = 0.0999$. The function $\phi(t)$ approximates the coordinates of the unpredictable solution $x(t)$ of the equation (2.2.13), as time increases. This can be shown in the same way as for function $\Theta(t)$. The 2-dimensional projection of the same solution on the $\phi_{11} - \phi_{12}$ plane and 3-dimensional projection on the $\phi_{11} - \phi_{12} - \phi_{23}$ space is shown in figure 13, respectively.

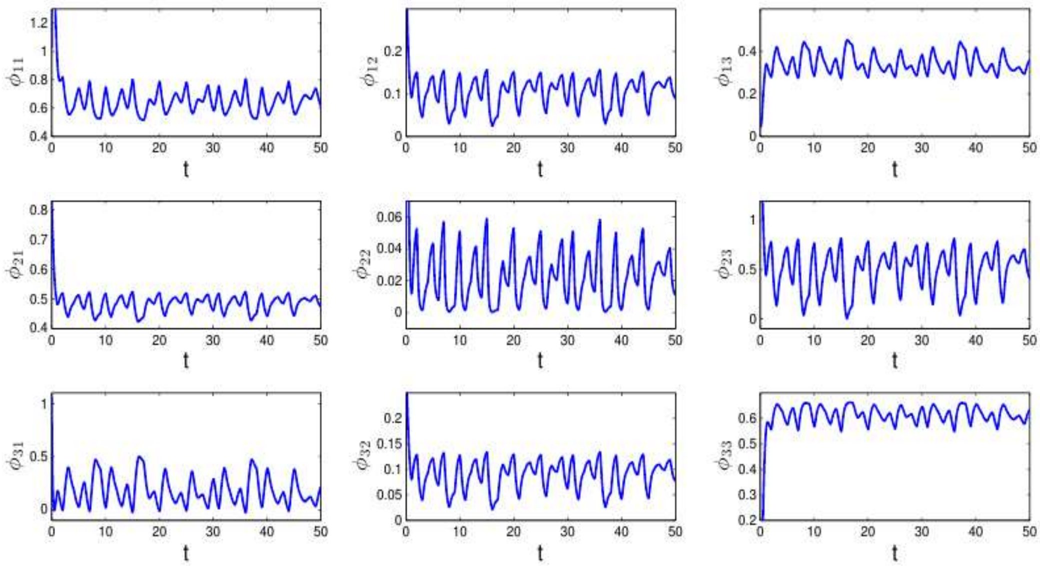

Figure 12 – The coordinates of the function $\phi(t)$, which exponentially converge to the coordinates of the strongly unpredictable solution of the SICNNs (2.3.13)

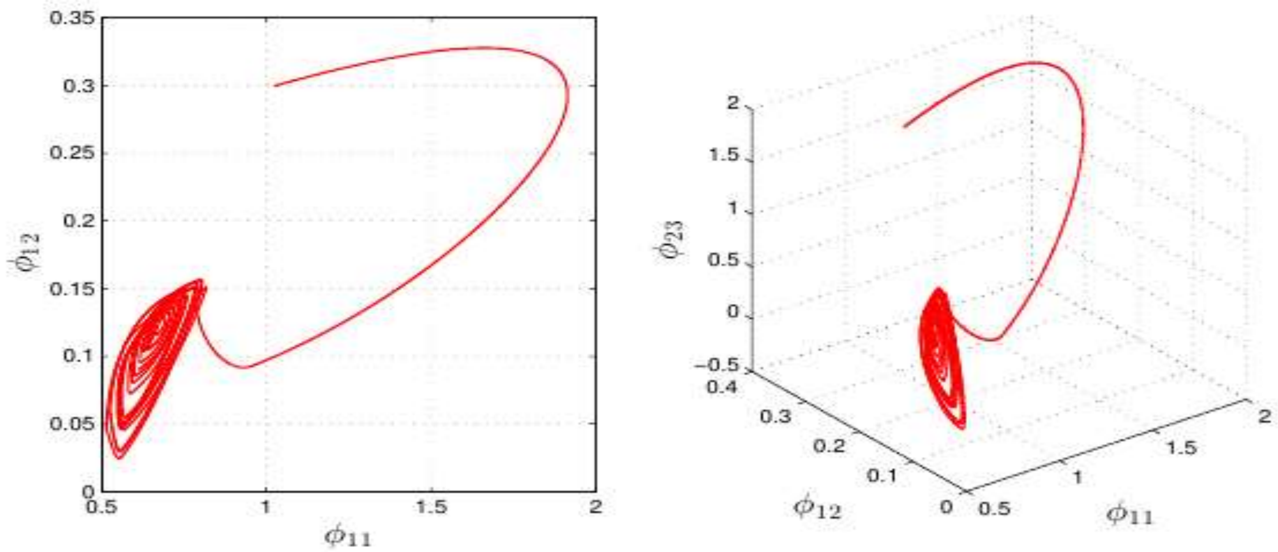

Figure 13 – The 2 and 3-dimensional projection of the trajectory of SICNNs (2.3.13) on the $\phi_{11} - \phi_{12}$ plane and $\phi_{11} - \phi_{12} - \phi_{23}$ space

2.3 Unpredictable oscillations of INNs

Most of the INN models are different from the traditional neural networks [85, p. 215; 95, 96] described by the first-order differential equations and it is more complex and more difficult to discuss their dynamical behaviors. Moreover, there exist significant backgrounds for investigating the inertial term in neural systems. For instance, the squid axon has a phenomenological inductance (Mauro, Conti, Dodge, and Schor, 1970) [97], the quasiactive membrane behavior of neurons can be modeled by adding inductance which makes the membrane to have electrical tuning, filtering behaviors (Koch, 1984) [98], the membrane of a hair cell in semicircular canals, can be implemented by equivalent circuits that contain an inductance (Angelaki and Correia, 1991) [99]. Therefore, some authors investigated neural networks by adding inertia. For, example, BAM neural networks [100-105], inertial CGNNs [106, 107], electronic neural networks with inertia [108], inertial memristive neural networks [109-111] have been studied.

Let us consider the following INNs:

$$\frac{d^2x_i(t)}{dt^2} = -a_i\frac{dx_i(t)}{dt} - b_ix_i(t) + \sum_{j=1}^{p} c_{ij}f_j(x_j(t)) + v_i(t), \qquad (2.3.1)$$

where $t, x_i \in \mathbb{R}, i = 1,2,\dots,p, p$ denotes the number of neurons in the network;

$x_i(t)$ with $i = 1,2,\dots,p,$ corresponds to the state of the unit i at time t; the second derivative is called an inertial term;

$b_i > 0, a_i > 0$ are constants;

f_i with $i = 1,2,\dots,p,$ denote the measures of activation to its incoming potentials of ith neuron;

c_{ij} for all $i, j = 1,2,\ldots,p$, are constants, which denote the synaptic connection weight of the unit j on the unit i;

$v_i(t)$ are external inputs on the ith neuron at time t.

We assume that the coefficients c_{ij} are real, the activation functions $f_i: \mathbb{R} \to \mathbb{R}$ are continuous functions satisfy the following condition:

$(I1)$ $|f_i(x_1) - f_i(x_2)| \le L_i|x_1 - x_2|$ for all $x_1, x_2 \in \mathbb{R}$, where $L_i > 0$ are Lipschitz constant, for all $i = 1,2,\ldots,p$, and $\max_{1\le i\le p} L_i = L$.

By introducing the following variable transformation:

$$y_i(t) = \xi_i \frac{dx_i(t)}{dt} + \zeta_i x_i(t), i = 1, \ldots, p, \qquad (2.3.2)$$

the neural network (2.3.1) can be written as:

$$\frac{dx_i(t)}{dt} = -\frac{\zeta_i}{\xi_i} x_i(t) + \frac{1}{\xi_i} y_i(t), \qquad (2.3.3)$$

$$\frac{dy_i(t)}{dt} = -\left(a_i - \frac{\zeta_i}{\xi_i}\right) y_i(t) - \left(\xi_i b_i - \zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right)\right) x_i(t) +$$

$$+\xi_i \sum_{j=1}^{p} c_{ij} f_j(x_j(t)) + \xi_i v_i(t), \qquad (2.3.4)$$

According to the results in [112], the couple $x(t) = \left(x_1(t), x_2(t), \ldots, x_p(t)\right)$, $\mathrm{y}(t) = (y_1(t), y_2(t), \ldots, y_p(t))$, is a bounded solution of (2.3.3), (2.3.4), if and only if the next integral equations are satisfied:

$$x_i(t) = \frac{1}{\xi_i} \int_{-\infty}^{t} e^{-\frac{\zeta_i}{\xi_i}(t-s)} y_i(t) ds, \qquad (2.3.5)$$

$$y_i(t) = \int_{-\infty}^{t} e^{-\left(a_i - \frac{\zeta_i}{\xi_i}\right)(t-s)} [(\xi_i b_i - \zeta_i(a_i - \frac{\zeta_i}{\xi_i}))x_i(t) + \xi_i \sum_{j=1}^{p} c_{ij} f_j\left(x_j(t)\right) +$$

$$+\xi_i v_i(t)]ds, \qquad (2.3.6)$$

where $i = 1, \ldots, p$. In what follows, we shall focus on the integral equations (2.4.5) and (2.3.6).

Denote by Σ the set of vector-functions, $\varphi(t) = (\varphi_1, \varphi_2, \ldots, \varphi_{2p})$, such that:

(K1) functions $\varphi(t)$ are uniformly continuous;

(*K2***)** there exists a positive number H such that $||\varphi||_1 < H$ for all $\varphi(t)$;

(*K3*) there exists a sequence, $t_n \to \infty$ as $n \to \infty$ such that $\varphi(t + t_n)$ uniformly converges to $\varphi(t)$ on each bounded interval of the real line.

Define on Σ an operator Π, such that:

$$\Pi\varphi(t) = (\Pi_1\varphi_1(t), \Pi_2\varphi_2(t), \ldots, \Pi_{2p}\varphi_{2p}(t))$$

and

$$\Pi_i\varphi_i(t) = \frac{1}{\xi_i}\int_{-\infty}^{t} e^{-\frac{\zeta_i}{\xi_i}(t-s)}\varphi_{i+p}(s)ds\,, \qquad i = 1,2,\ldots,p, \tag{2.3.7}$$

and

$$\Pi_i\varphi_i(t) = \int_{-\infty}^{t} e^{-\left(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}\right)(t-s)}\left[\left(\xi_{i-p}b_{i-p} - \zeta_{i-p}\left(a_{i-p} - \frac{\zeta_{i-p}}{\xi_{i-p}}\right)\right)\varphi_{i-p}(t) + \right.$$

$$\left. +\xi_{i-p}\sum_{j=1}^{p} c_{(i-p)j}f_j(\varphi_j(t)) + \xi_{i-p}v_{i-p}(t)\right]ds, \qquad i = p+1, \ldots, 2p. \tag{2.3.8}$$

The following conditions will be needed throughout the paper:

$(I2)$ the functions $v_i(t), i = 1, \ldots, p,$ in system (2.3.1) are unpredictable, they belong to Σ and there exist positive numbers ε_0, δ and the sequence $s_n \to \infty$ as $n \to \infty$, such that $|v_i(t + t_n) - v_i(t)| \geq \varepsilon_0$ for all $t \in [s_n - \delta, s_n + \delta], i = 1, \ldots, p,$ and $n \in \mathbb{N}$;

$(I3)$ there exists a positive number M_f such that $|f_i(s)| \leq M_f$, $i = 1, \ldots, p, |s| < H.$

Moreover, we assume that for positive real numbers ζ_i and ξ_i, $i = 1, \ldots, p$ the following inequalities are valid:

$(I4)$ $a_i > \frac{\zeta_i}{\xi_i} + \xi_i, \zeta_i > \xi_i > 1;$

$(I5)$ $(a_i - \frac{\zeta_i}{\xi_i}) - (|\zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + \xi_i) > 0;$

$(I6)$ $\frac{\xi_i M_f \sum_{j=1}^{p} c_{ij}}{(a_i - \frac{\zeta_i}{\xi_i}) - (|\zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + \xi_i)} < H;$

$(I7)$ $\frac{1}{(a_i - \frac{\zeta_i}{\xi_i})}(|\zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + L\xi_i\sum_{j=1}^{p} c_{ij}) < 1;$

$(I8)\ \max_i(\frac{1}{\xi_i}, |\zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + L\xi_i \sum_{j=1}^{p} c_{ij}) < \min_i(\frac{\zeta_i}{\xi_i}, a_i - \frac{\zeta_i}{\xi_i}).$

Lemma 2.3.1. The operator Π is invariant in Σ.

Proof. For a function $\varphi(t) \in \Sigma$ and fixed $i = 1,\dots,2p$, we have that

$$|\Pi_i\varphi_i(t)| = |\frac{1}{\xi_i}\int_{-\infty}^{t} e^{-\frac{\zeta_i}{\xi_i}(t-s)}\varphi_{i+p}(s)ds\,| \le \frac{1}{\zeta_i}|\varphi_{i+p}(t)| \le \frac{H}{\zeta_i},$$

for $i = 1,\dots,p$, and

$$|\Pi_i\varphi_i(t)| = |\int_{-\infty}^{t} e^{-\left(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}\right)(t-s)}\,[\left(\xi_{i-p}b_{i-p} - \zeta_{i-p}\left(a_{i-p} - \frac{\zeta_{i-p}}{\xi_{i-p}}\right)\right)\varphi_{i-p}(t) +$$

$$+\xi_{i-p}\sum_{j=1}^{p} c_{(i-p)j}f_j\left(\varphi_j(t)\right) + \xi_{i-p}v_{i-p}(t)]ds| \le$$

$$\le \int_{-\infty}^{t} e^{-\left(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}\right)(t-s)}\,[|\xi_{i-p}b_{i-p} - \zeta_{i-p}\left(a_{i-p} - \frac{\zeta_{i-p}}{\xi_{i-p}}\right)|H +$$

$$+\xi_{i-p}\sum_{j=1}^{p} c_{(i-p)j}M_f + \xi_{i-p}H]ds \le$$

$$\le \frac{1}{(a_i - \frac{\zeta_i}{\xi_i})}\left[H(|\xi_{i-p}b_{i-p} - \zeta_{i-p}\left(a_{i-p} - \frac{\zeta_{i-p}}{\xi_{i-p}}\right)| + \xi_{i-p} + \xi_{i-p}\sum_{j=1}^{p} c_{(i-p)j}M_f)\right],$$

for $i = p+1,\dots,2p$.

Conditions (I5), (I6) imply that $|\Pi_i\varphi_i(t)| < H$, for each $i = 1,\dots,2p$. So that $||\Pi\varphi||_1 = \max_i |\Pi_i\varphi_i| < H$. Thus, condition (K2) is valid.

Let us fix a positive number ε and a section $[a,b], -\infty < a < b < \infty$. We will show that for sufficiently large n it is true that $||\Pi\varphi(t+t_n) - \Pi\varphi(t)||_1 < \varepsilon$ on $[a,b]$. One can find that

$$|\Pi_i\varphi_i(t+t_n) - \Pi_i\varphi_i(t)| = |\frac{1}{\xi_i}\int_{-\infty}^{t} e^{-\frac{\zeta_i}{\xi_i}(t-s)}(\varphi_{i+p}(s+t_n) - \varphi_{i+p}(s))ds\,|,$$

for $i = 1,\dots,p$, and

$$|\Pi_i\varphi_i(t+t_n)-\Pi_i\varphi_i(t)|=$$

$$=|\int_{-\infty}^{t} e^{-\left(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}\right)(t-s)}[(\xi_{i-p}b_{i-p}-\zeta_{i-p}(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}))(\varphi_{i-p}(s+t_n)-\varphi_{i-p}(s))$$

$$+\xi_{i-p}\sum_{j=1}^{p} c_{(i-p)j}\left(f_j\left(\varphi_j(s+t_n)\right)-f_j\left(\varphi_j(s)\right)\right)+$$

$$+\xi_{i-p}(v_{i-p}(s+t_n)-v_{i-p}(s))]ds|,$$

for $i=p+1,\ldots,2p$.
Choose numbers $c<a$ and $\xi>0$, satisfying the following inequalities:

$$\frac{2H}{\zeta_i}e^{-\frac{\zeta_i}{\xi_i}(a-c)}<\frac{\varepsilon}{2}, \tag{2.3.9}$$

$$\frac{2H}{a_i-\frac{\zeta_i}{\xi_i}}(|\zeta_i(a_i-\frac{\zeta_i}{\xi_i})-\xi_i b_i|+L\xi_i\sum_{j=1}^{p}c_{ij}+\xi_i)e^{-(a_i-\frac{\zeta_i}{\xi_i})(a-c)}<\frac{\varepsilon}{2}, \tag{2.3.10}$$

$$\frac{\xi}{\zeta_i}<\frac{\varepsilon}{2}, \tag{2.3.11}$$

$$\frac{\xi}{(a_i-\frac{\zeta_i}{\xi_i})}(|\zeta_i\left(a_i-\frac{\zeta_i}{\xi_i}\right)-\xi_i b_i|+L\xi_i\textstyle\sum_{j=1}^{p}c_{ij})<\frac{\varepsilon}{2}, \tag{2.3.12}$$

Consider the number n sufficiently large such that $|\varphi_i(t+t_n)-\varphi_i(t)|<\xi$, $i=1,\ldots,2p$, and $|v_i(t+t_n)-v_i(t)|<\xi$, $i=1,\ldots,p$, on $t\in[c,b]$. Then, for all $t\in[a,b]$, it is true that:

$$|\Pi_i\varphi_i(t+t_n)-\Pi_i\varphi_i(t)|\le|\frac{1}{\xi_i}\int_{-\infty}^{c} e^{-\frac{\zeta_i}{\xi_i}(t-s)}\left(\varphi_{i+p}(s+t_n)-\varphi_{i+p}(s)\right)ds\,|+$$

$$+|\frac{1}{\xi_i}\int_{c}^{t} e^{-\frac{\zeta_i}{\xi_i}(t-s)}(\varphi_{i+p}(s+t_n)-\varphi_{i+p}(s))ds\,|\le\frac{2H}{\zeta_i}e^{-\frac{\zeta_i}{\xi_i}(a-c)}+\frac{\xi}{\zeta_i},$$

for $i=1,\ldots,p$, and

$$|\Pi_i\varphi_i(t+t_n)-\Pi_i\varphi_i(t)|\le$$

$$\leq |\int_{-\infty}^{c} e^{-\left(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}\right)(t-s)}\,[(\xi_{i-p}b_{i-p}-\zeta_{i-p}(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}))(\varphi_{i-p}(s+t_n)-\varphi_{i-p}(s))$$

$$+\xi_{i-p}\sum_{j=1}^{p} c_{(i-p)j}\left(f_j\left(\varphi_j(s+t_n)\right)-f_j\left(\varphi_j(s)\right)\right)+$$

$$+\xi_{i-p}(v_{i-p}(s+t_n)-v_{i-p}(s))]ds|+$$

$$+|\int_{c}^{t} e^{-\left(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}\right)(t-s)}\,[(\xi_{i-p}b_{i-p}-\zeta_{i-p}(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}))(\varphi_{i-p}(s+t_n)-\varphi_{i-p}(s))$$

$$+\xi_{i-p}\sum_{j=1}^{p} c_{(i-p)j}(f_j(\varphi_j(s+t_n))-f_j(\varphi_j(s)))+\xi_{i-p}(v_{i-p}(s+t_n)v_{i-p}(s))]ds|\leq$$

$$\leq \frac{1}{a_i-\frac{\zeta_i}{\xi_i}}(2H\left|\zeta_i(a_i-\frac{\zeta_i}{\xi_i})-\xi_i b_i\right|+2HL\xi_i\sum_{j=1}^{p} c_{ij}+2H\xi_i)e^{-\left(a_i-\frac{\zeta_i}{\xi_i}\right)(a-c)}+$$

$$+\frac{1}{(a_i-\frac{\zeta_i}{\xi_i})}(|\zeta_i\left(a_i-\frac{\zeta_i}{\xi_i}\right)-\xi_i b_i|\xi+L\xi_i\sum_{j=1}^{p} c_{ij}\,\xi),$$

for $i=p+1,\ldots,2p$.

The inequalities (2.3.9) -(2.3.12) imply that $||\Pi\varphi(t+t_n)-\Pi\varphi(t)||_1<\varepsilon$, for $t\in[a,b]$. Since ε is arbitrary small number, the condition (K3) is satisfied. Condition (K1) follows from the boundedness of its derivative. The lemma is proved.

Lemma 2.3.2. The operator Π is a contraction mapping on Σ.

Proof. For $u,v\in\Sigma$, one can attain that:

$$|\Pi_i u_i(t)-\Pi_i v_i(t)|\leq$$
$$\leq |\frac{1}{\xi_i}\int_{-\infty}^{t} e^{-\frac{\zeta_i}{\xi_i}(t-s)}\left(u_{i+p}(s)-v_{i+p}(s)\right)ds\leq\frac{1}{\zeta_i}||u(t)-v(t)||_1,$$

$i=1,\ldots,p$, and

$$|\Pi_i u_i(t)-\Pi_i v_i(t)|\leq$$

$$\leq \left| \int_{-\infty}^{c} e^{-\left(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}\right)(t-s)} [(\xi_{i-p}b_{i-p} - \zeta_{i-p}(a_{i-p} - \frac{\zeta_{i-p}}{\xi_{i-p}}))(u_{i-p}(s) - v_{i-p}(s)) + \right.$$

$$\left. +\xi_{i-p} \sum_{j=1}^{p} c_{(i-p)j}(f_j\left(u_j(s)\right) - f_j\left(v_j(s)\right))]ds \right| \leq$$

$$\leq \frac{1}{(a_i - \frac{\zeta_i}{\xi_i})} (|\zeta_{i-p}(a_{i-p} - \frac{\zeta_{i-p}}{\xi_{i-p}}) - \xi_{i-p}b_{i-p}| + L\xi_{i-p} \sum_{j=1}^{p} c_{(i-p)j})|u(t) - v(t)||_1,$$

$i = p + 1, \ldots, 2p$.

The last inequalities yield $||\Pi u - \Pi v||_1 = \max_i(\frac{1}{\xi_i}, \frac{1}{(a_i-\frac{\zeta_i}{\xi_i})}(|\zeta_i\left(a_i - \frac{\zeta_i}{\xi_i}\right) - \xi_i b_i| + +L\xi_i \sum_{j=1}^{p} c_{ij}))||u - v||_1$. Hence, in accordance with conditions (I4), (I7) the operator Π is contractive.

Theorem 2.3.1. Assume that conditions (I1) -(I8) are fulfilled. Then the system (2.3.1) admits a unique asymptotically stable unpredictable solution.

Proof. Let us show that the space Σ is complete. Consider a Cauchy sequence $\phi_k(t)$ in Σ, which converges to a limit function $\phi(t)$ on $\mathbb{R}$. It suffices to show that $\phi(t)$ satisfies condition (K3), since other two conditions can be easily checked. Fix a closed and bounded interval $I \subset \mathbb{R}$. We have that:

$$||\phi(t + t_n) - \phi(t)|| \leq ||\phi(t + t_n) - \phi_k(t + t_n)|| + ||\phi_k(t + t_n)|| - \phi_k(t)|| +$$
$$+||\phi_k(t) - \phi(t)|| \qquad (2.3.13)$$

Now, one can take sufficiently large n and k such that each term on right hand-side of (2.3.11) is smaller than $\frac{\varepsilon}{3}$ for an arbitrary positive ε and $t \in I$. The inequality implies that $||\phi\left(t + t_p\right) - \phi(t)|| \leq \varepsilon$ on *I*. That is the sequence $\phi(t + t_n)$ uniformly converges to $\phi(t)$ on *I*. The completeness of Σ is proved. By the contractive mapping theorem, duo to Lemmas 2.3.1 and 2.3.2, there exists a unique solution $\omega(t) \in \Sigma$ of the equation (2.3.1).

Next, we prove the unpredictability property for $\omega(t)$. It is true that:

$$\omega_i(t + t_n) - \omega_i(t) = \omega_i(s_n + t_n) - \omega_i(s_n) - \int_{s_n}^{t} \frac{\zeta_i}{\xi_i}\left(\omega_i(s + t_n) - \omega_i(s)\right)ds +$$

$$+\int_{s_n}^{t}\frac{1}{\xi_i}\Big(\omega_{i+p}(s+t_n)-\omega_{i+p}(s)\Big)ds,\ \text{for } i=1,\dots,p, \tag{2.3.14}$$

and

$$\omega_i(t+t_n)-\omega_i(t)=\omega_i(s_n+t_n)-\omega_i(s_n)-$$

$$-\int_{s_n}^{t}\left(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}\right)\big(\omega_i(s+t_n)-\omega_i(s)\big)ds-$$

$$-\int_{s_n}^{t}\left(\zeta_{i-p}\left(a_{i-p}-\frac{\zeta_{i-p}}{\xi_{i-p}}\right)-\xi_{i-p}b_{i-p}\right)\Big(\omega_{i-p}(s+t_n)-\omega_{i-p}(s)\Big)ds+$$

$$+\int_{s_n}^{t}\xi_{i-p}\sum_{j=1}^{p}c_{(i-p)j}(f_j\left(\omega_j(s+t_n)\right)-f_j\left(\omega_j(s)\right))ds+$$

$$+\int_{s_n}^{t}\xi_{i-p}\left(\upsilon_{i-p}(s+t_n)-\upsilon_{i-p}(s)\right)ds,\ \text{for } i=p+1,\dots,2p. \tag{2.3.15}$$

There exist a positive number κ and and integer l such that the following inequalities are valid for all $i=1,\dots,p$:

$$\kappa<\delta, \tag{2.3.16}$$

$$\xi_i\kappa\geq\frac{3}{2l}\kappa\left[(a_i-\frac{\zeta_i}{\xi_i})+|\zeta_i(a_i-\frac{\zeta_i}{\xi_i})-\xi_ib_i|+L\xi_i\sum_{j=1}^{p}c_{ij}\right]+\kappa\frac{1}{l}+\frac{1}{2l}, \tag{2.3.17}$$

$$\frac{\kappa}{\xi_i}\geq\frac{3}{l}\left(\kappa\frac{\zeta_i}{\xi_i}+1\right), \tag{2.3.18}$$

and

$$\omega_i(t+s)-\omega_i(t)<\frac{\varepsilon_0}{4l^2},\ \text{for } i=1,2,\dots,2p,\ t\in\mathbb{R},|s|<\kappa, \tag{2.3.19}$$

where δ satisfies to condition (I2). Let the numbers κ and l as well as number $n\in\mathbb{N}$ be fixed.

Let us, first, fix one of the coordinates $i = p+1, p+2, \dots, 2p$. The proof naturally falls into two parts: (i) $\Delta_1 = |\omega_i(t_n + s_n) - \omega_i(s_n)| \geq \varepsilon_0/l$ and (ii) $\Delta_1 = |\omega_i(t_n + s_n) - \omega_i(s_n)| < \varepsilon_0/l$.

(i) For the case $\Delta_1 \geq \varepsilon_0/l$, by (2.3.19) we get that:

$$|\omega_i(t + t_n) - \omega_i(t)| \geq |\omega_i(t_n + s_n) - \omega_i(s_n)| - |\omega_i(s_n) - \omega_i(t)| -$$

$$-|\omega_i(t + t_n) - \omega_i(t_n + s_n)| \geq \frac{\varepsilon_0}{l} - \frac{\varepsilon_0}{4l^2} - \frac{\varepsilon_0}{4l^2} \geq \frac{\varepsilon_0}{l} - \frac{\varepsilon_0}{2l^2} \geq \frac{\varepsilon_0}{2l}, \quad (2.3.20)$$

if $t \in [s_n - \kappa, s_n + \kappa]$, $i = p+1, p+2, \dots, 2p$, and $n \in \mathbb{N}$.

(ii) From (2.3.19) it follows that:

$$|\omega_i(t + t_n) - \omega_i(t)| \leq |\omega_i(t_n + s_n) - \omega_i(s_n)| + |\omega_i(s_n) - \omega_i(t)| +$$

$$+|\omega_i(t + t_n) - \omega_i(t_n + s_n)| \leq \frac{\varepsilon_0}{l} + \frac{\varepsilon_0}{4l^2} + \frac{\varepsilon_0}{4l^2} < \frac{3\varepsilon_0}{2l}, \; i = 1,2,\dots,2p, \quad (2.3.21)$$

if $t \in [s_n, s_n + \kappa]$.

Now, using (2.3.16), (2.3.21) and relation (2.3.15), we have that:

$$|\omega_i(t + t_n) - \omega_i(t)| \geq \Big| \int_{s_n}^{t} \xi_{i-p}\left(v_{i-p}(s + t_n) - v_{i-p}(s)\right) ds -$$

$$-\Big| \int_{s_n}^{t} \left(a_{i-p} - \frac{\zeta_{i-p}}{\xi_{i-p}}\right)\left(\omega_i(s + t_n) - \omega_i(s)\right) ds \Big| -$$

$$-\Big| \int_{s_n}^{t} \left(\zeta_{i-p}\left(a_{i-p} - \frac{\zeta_{i-p}}{\xi_{i-p}}\right) - \xi_{i-p} b_{i-p}\right)\left(\omega_{i-p}(s + t_n) - \omega_{i-p}(s)\right) ds \Big| -$$

$$-\Big| \int_{s_n}^{t} \xi_{i-p} \sum_{j=1}^{p} c_{(i-p)j}\left(f_j\left(\omega_j(s + t_n)\right) - f_j\left(\omega_j(s)\right)\right) ds \Big| -$$

$$-|\omega_i(s_n + t_n) - \omega_i(s_n)| \geq$$

$$\geq \xi_{i-p}\varepsilon_0\kappa - \frac{3\varepsilon_0}{2l}\kappa\left(a_{i-p} - \frac{\zeta_{i-p}}{\xi_{i-p}}\right) + \left|\zeta_{i-p}\left(a_{i-p} - \frac{\zeta_{i-p}}{\xi_{i-p}}\right) - \xi_{i-p} b_{i-p}\right| +$$

$$+L\xi_{i-p}\sum_{j=1}^{p}c_{(i-p)j}-\kappa\frac{\varepsilon_0}{l}, \tag{2.4.22}$$

for $t\in[s_n,s_n+\kappa]$ and $i=p+1,p+2,\ldots,2p$. According to (2.3.17), from (2.3.20) and (2.3.22) we got that:

$$|\omega_i(t+t_n)-\omega_i(t)|\geq\frac{\varepsilon_0}{2l},\text{for } t\in[s_n,s_n+\kappa], i=p+1,p+2,\ldots,2p. \tag{2.3.23}$$

Now, we show unpredictability for $\omega_i(t), i=1,2,\ldots,p$. Similarly to the coordinates $i=p+1,p+2,\ldots,2p$, let us consider two cases: (i) $\Delta_2=|\omega_i(t_n+ +s_n)-\omega_i(s_n)|\geq\varepsilon_0/l^2$ and (ii) $\Delta_2=|\omega_i(t_n+s_n)-\omega_i(s_n)|<\varepsilon_0/l^2$.

(i) For $\Delta_2\geq\varepsilon_0/l^2$, by (2.3.19) we get that:

$$|\omega_i(t+t_n)-\omega_i(t)|\geq|\omega_i(t_n+s_n)-\omega_i(s_n)|-|\omega_i(s_n)-\omega_i(t)|-$$

$$-|\omega_i(t+t_n)-\omega_i(t_n+s_n)|\geq\frac{\varepsilon_0}{l}-\frac{\varepsilon_0}{4l^2}-\frac{\varepsilon_0}{4l^2}=\frac{\varepsilon_0}{2l^2}, \tag{2.3.24}$$

if $t\in[s_n-\kappa,s_n+\kappa]$, $i=1,2,\ldots,2p$, and $n\in\mathbb{N}$.

(ii) If $\Delta_2<\varepsilon_0/l^2$, then from (2.3.15) and (2.3.23) it follows that:

$$|\omega_i(t+t_n)-\omega_i(t)|\geq\left|\int_{s_n}^{t}\frac{1}{\xi_i}\Big(\omega_{i+p}(s+t_n)-\omega_{i+p}(s)\Big)\,ds\right|-$$

$$-|\omega_i(s_n+t_n)-\omega_i(s_n)|-\left|\int_{s_n}^{t}\frac{\zeta_i}{\xi_i}\big(\omega_i(s+t_n)-\omega_i(s)\big)ds\right|\geq$$

$$\geq\frac{1}{\xi_i}\kappa\frac{\varepsilon_0}{2l}-\frac{\varepsilon_0}{l^2}-\frac{\zeta_i}{\xi_i}\kappa\frac{3\varepsilon_0}{2l^2}, \tag{2.3.25}$$

if $t\in[s_n,s_n+\kappa]$, $i=1,2,\ldots,p$. By inequality (2.3.18) we obtain that $|\omega_i(t+t_n)-\omega_i(t)|\geq\frac{\varepsilon_0}{2l^2}$, for $t\in[s_n,s_n+\kappa]$, $i=1,2,\ldots,p$. Thus, one can conclude that $\omega(t)$ is an unpredictable solution with $\bar{\varepsilon_0}=\frac{\varepsilon_0}{2l^2}$, $\bar{s_n}=s_n+\frac{3\kappa}{4}$ and $\bar{\delta}=\delta+\frac{\kappa}{4}$.

Finally, we discuss the stability of the solution $\omega(t)$. Let us define 2p-dimensional function $z(t)=(x_1(t),\ldots,x_p(t),y_1(t),\ldots,y_p(t))$, and rewrite system (2.3.3), (2.3.4) in the vector form:

$$\frac{dz(t)}{dt}=Az(t)+F(t,z), \tag{2.3.26}$$

where matrix $A = diag(-\frac{\zeta_1}{\xi_1}, \ldots, -\frac{\zeta_p}{\xi_p}, -(a_1 - \frac{\zeta_1}{\xi_1}), \ldots, -(a_p - \frac{\zeta_p}{\xi_p}))$ and $F(t,z) = (F_1(t,z), F_2(t,z), \ldots, F_{2p}(t,z))$ is a vector-function, such that:

$$F_i(t,z) = \frac{1}{\xi_i} z_{i+p}(t), \qquad i = 1, \ldots, p,$$

and

$$F_i(t,z) = -\left(\xi_{i-p} b_{i-p} - \zeta_{i-p}\left(a_{i-p} - \frac{\zeta_{i-p}}{\xi_{i-p}}\right)\right) z_{i-p}(t) +$$

$$+\xi_{i-p} \sum_{j=1}^{p} c_{(i-p)j} f_j\left(z_j(t)\right) + \xi_{i-p} v_{i-p}(t), \qquad i = p+1, \ldots, 2p.$$

We denote $\lambda = \min_i(\frac{\zeta_i}{\xi_i}, a_i - \frac{\zeta_i}{\xi_i})$, and $L_F = \max_i(\frac{1}{\xi_i}, |\zeta_i(a_i - \frac{\zeta_i}{\xi_i}) - \xi_i b_i| +$
$+L\xi_i \sum_{j=1}^{p} c_{ij})$, $i = 1, \ldots, p$.
Consider

$$\omega(t) = e^{A(t-t_o)}\omega(t_o) + \int_{t_o}^{t} e^{A(t-s)} F(s, \omega(s)) ds,$$

and another solution $\psi(t)$ of the system (2.3.26)

$$\psi(t) = e^{A(t-t_o)}\psi(t_o) + \int_{t_o}^{t} e^{A(t-s)} F(s, \psi(s)) ds.$$

We have that

$$|||\omega(t) - \psi(t)||_1 \leq$$

$$\leq e^{-\lambda(t-t_o)}||\omega(t_o) - \psi(t_o)||_1 + \int_{t_o}^{t} e^{-\lambda(t-s)} L_F ||\omega(s) - \psi(s)||_1 ds,$$

for $t \geq t_o$.

Applying Gronwall-Bellman Lemma, one can attain that:

$$||\omega(t) - \psi(t)||_1 \leq ||\omega(t_o) - \psi(t_o)||_1 e^{(L_F - \lambda)(t-t_o)}, t \geq t_o. \quad (2.3.27)$$

Condition (I8) implies that $\omega(t)$ is uniformly exponentially stable solution of the system (2.3.26).

Example 9. Let us take into account the system:

$$\frac{d^2x_i(t)}{dt^2} = -a_i\frac{dx_i(t)}{dt} - b_ix_i(t) + \sum_{j=1}^{3} c_{ij}f_j(x_j(t)) + v_i(t), \tag{2.3.28}$$

$i = 1,2,3, a_1 = 6, a_2 = 5, a_3 = 7, b_1 = 8, b_2 = 6, b_3 = 8, f(x) = 0.35arctg(x),$

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \begin{pmatrix} 0.02 & 0.03 & 0.02 \\ 0.04 & 0.05 & 0.01 \\ 0.03 & 0.06 & 0.02 \end{pmatrix}$$

and $v_1(t) = -58\Theta^3(t) + 5, v_2(t) = 76\Theta^3(t) + 4, v_3(t) = 42\Theta(t) - 3, \Theta(t) =$ $= \int_{-\infty}^{t} e^{-3(t-s)}\Omega(s)ds$. The function $v(t)$ is unpredictable in accordance with properties of unpredictable functions. The conditions (I1) -(I8) hold for the network (2.3.28) with $\xi_1 = \xi_2 = 2, \xi_3 = 3, \zeta_1 = \zeta_2 = 4, \zeta_3 = 4.4, L = 0.35, M_f = 0.56,$ $H = 2$. It is not difficult to calculate that $L_F = 0.57$ and $\lambda = 1.47$. Consequently, there exists the unpredictable solution, $x_i(t)$, of the system. Since it is not possible to indicate the initial value of the solution, we apply the property of asymptotic stability, since any solution from the domain ultimately approaches the unpredictable oscillation. That is, to visualize the behavior of the unpredictable oscillation, we consider the simulation of another solution. We shall simulate the solution $\omega(t)$ with initial conditions $\omega_1(0) = 1.023, \omega_2(0) = 1.516, \omega_3 = 0.275$.

Utilizing (2.3.27), we have that:

$$||\varphi(t) - \omega(t)||_1 \leq ||\varphi(0) - \omega(0)||_1 e^{(L_F-\lambda)t} < 2He^{-0.9t} \leq 4e^{-0.9t}, t > 0.$$

Thus, if $t > \frac{9}{10}(5ln10 + ln4) \approx 11.77$, then $||\varphi(t) - \omega(t)||_1 < 10^{-5}$, and we can say that the graphs of the functions match visually. In other words, what is seen in figures 14 and 15 for a sufficiently large time can be accepted as parts of the graph and trajectory of the unpredictable solution. Both of the figures reveal the irregular behavior of system (2.3.28).

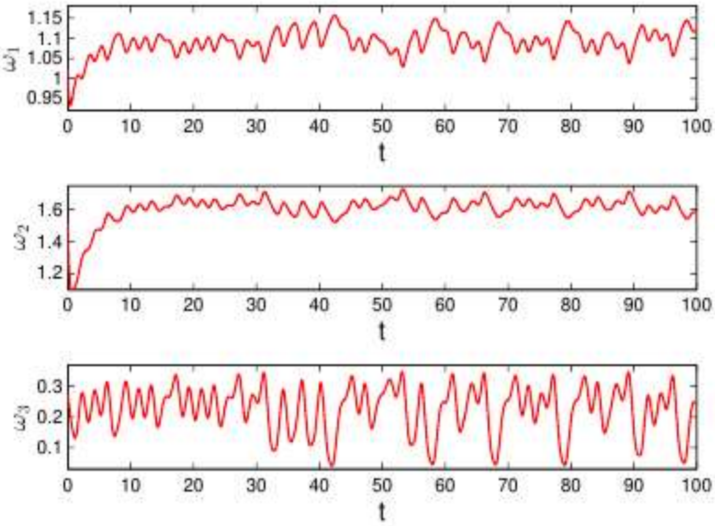

Figure 14 – The coordinates of the function $\omega(t)$, which exponentially converge to the coordinates of the strongly unpredictable solution of the system (2.3.28)

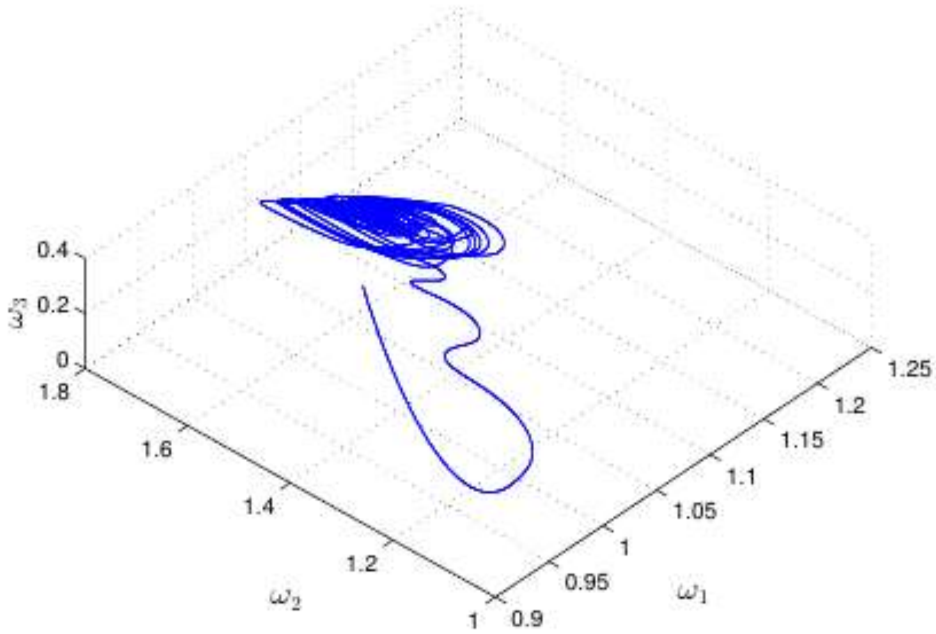

Figure 15 – The irregular trajectory of the function $\omega(t)$

CONCLUSION

The thesis is devoted to the study of unpredictable oscillations of differential equations and neural networks.

In the thesis the definitions of unpredictable sequences and unpredictable functions are presented. An example of an unpredictable function is constructed.

Main results:

– the existence and uniqueness of uniformly asymptotically stable unpredictable solution of linear differential equations;

– the conditions for the existence and uniqueness of uniformly exponentially stable unpredictable and unpredictable solution of quasilinear differential equations;

– the existence and uniqueness of an asymptotically stable unpredictable and strongly unpredictable solution of SICNNs;

– the existence and uniqueness of an asymptotically stable unpredictable solution of INNs;

– construction of unpredictable functions.

In each section, examples with numerical simulations are presented to illustrate the feasibility of the obtained results.

The concept of unpredictable oscillations can be useful for finding more delicate features in systems with complicated dynamics. In this framework, the results can be developed for partial differential equations, integrodifferential equations, functional differential equations, evolution systems, and neural networks.

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REFERENCES

1 Akhmet M., Fen M.O. Unpredictable points and chaos // Communications in Nonlinear Science and Numerical Simulation. – 2016. – №40. – P. 1-5.

2 Akhmet M., Fen M.O. Existence of unpredictable solutions and chaos // Turkish Journal of Mathematics. – 2017. – Vol. 41, №2. – P. 254-266.

3 Akhmet M., Fen M.O. Poincare chaos and unpredictable functions // Communications in Nonlinear Science and Numerical Simulation. – 2017. – №48. – P. 85-94.

4 Akhmet M., Fen M. Non-autonomous equations with unpredictable solutions // Communications in Nonlinear Sciences and Numerical Simulation. – 2018. – №59. – P. 657-670.

5 Andronov A.A., Vitt A.A., Khaikin S.E. Theory of oscillations. – Oxford: Pergamon Press, 1966. – 815 p.

6 Malkin I.G. Some problems in the theory of non-linear oscillations. – M.; L., 1956. – 173 p.

7 Bogolyubov N.N., Mitropol'skii Yu.A. Asymptotic methods in the theory of non-linear oscillations. – Delhi: Hindustan Publication Company, 1961. – 548 p.

8 Hayashi C. Nonlinear oscillations in physical systems. – NY.: McGraw-Hill, 1964. – 392 p.

9 Minorski N. Nonlinear oscillations. – Princeton: Van Nostrand, 1962. – 714 p.

10 Nayfeh A.H., Mook D.T. Nonlinear oscillations. – NY.: Wiley, 1979. – 720 p.

11 Schmidt G., Tondl A. Non-linear vibrations. – Cambridge: Cambridge University Press, 1967. – 420 p.

12 Urabe M. Nonlinear autonomous oscillations. Analytical theory – NY.: Academic Press, 1967. – 320 p.

13 Farkas M. Periodic Motion. – NY.: Springer-Verlag, 1994. – 577 pp.

14 Nemytskii V.V., Stepanov V.V. Qualitative Theory of Differential Equations. – Princeton; NJ, USA: Princeton University Press, 1960. – 523 p.

15 Corduneanu C. Almost Periodic Oscillations and Waves. – NY.: Springer-Verlag, 2009. – 313 p.

16 Fink A.M. Almost periodic differential equations. – NY.: Springer-Verlag, 1974. – 342 p.

17 Levitan B.M., Zhikov V.V. Almost periodic functions and differential equations. – Cambridge: Cambridge University Press, 1983. – 224 p.

18 Bener P.R. Recurrent solutions to systems of ordinary differential equations // Journal of Differential Equations. – 1969. – №5. – P. 271-282.

19 Pokrovskii A.V., Abodayeh K., McInerney J. Recurrent oscillations in systems with hysteresis nonlinearities // Physica B: Condensed Matter. – 2001. – Vol. 306, Issues 1-4. – P. 191-194.

20 Sell G.R. Topological Dynamics and Ordinary Differential Equations. – London, UK: Van Nostrand Reinhold Company, 1971. – 199 p.

21 Hino Y. Reccurent solutions for linear almost periodic systems // Funkcialaj Ekvacioj. – 1985. – №28. – P. 117-119.

22 Kumar A., Bhagat R.P. Poisson stability in product of dynamical systems // International Journal of Mathematics and Mathematical Sciences. – 1987. – Vol. 10(3). – P. 613-614.

23 Knight R.A. Recurrent and Poisson stable flows // Proceedings of the American mathematical society. – 1981. – Vol. 83(1). – P. 49-53.

24 Shcherbakov B.A., Cheban D. Poisson asymptotic stability of motions of dynamical systems and their comparability with regard to the recurrence property in the limit // Differential Equations. – 1997. – Vol. 13(5). – P. 898-906.

25 Poincaré H. Les methodes nouvelles de la mecanique celeste. – Paris: Gauthier-Villars, 1892. – Vol. 1. – 408 p.

26 Poincaré H. Les methodes nouvelles de la mecanique celeste. – Paris, 1899. – Vol. 3. – 426 p.

27 Lyapunov A.M. Stability of Motion. – NY.; London: Academic Press, 1966. – 203 p.

28 Bohl P. Ueber einige Differentialgleichungen allgemeinen Charakters, welche in der Mechanik anwendbar sind. – Dorpat, 1900. – 113 p.

29 Bohl P. Über eine Differentialgleichung der Störungstheorie // Grelles Journal. – 1906. – №131. – P. 268-321.

30 Esclangon E. Les fonctions quasi-périodiques. – Paris: Gauthier-Villars, 1904. – 306 p.

31 Bohr H. Zur Theorie der fastperiodischen Funktionen I // Acta Mathematica. – 1925. – №8. – P. 29-127.

32 Bohr H. Zur Theorie der fastperiodischen Funktionen II // Acta Mathematica. – 1925. – №45. – P. 101-214.

33 Bohr H. Almost-Periodic Functions. – NY.: Chelsea Publishing Company, 1951. – 114 p.

34 Bogolyubov N.N. On some arithmetic properties of almost periods. – Kiev: Akademiya Nauk Ukrainian SSR, 1939. – 185 p.

35 Besicovitch A.S. Almost Periodic Functions. – Cambridge: Dover, 1954. – 179 p.

36 Bohner S. A new approach to almost periodicity // Proceedings of the National Academy of Science, USA. – 1962. – №45. – P. 195-205.

37 Stepanov V.V. Sur quelques géneralisations des fonctions presque périodiques // Comptes rendus de l'Académie des Sciences. – 1925. – №181. – P. 90-92.

38 Bohner S. Abstrakte fastperiodische Funktionen // Acta Mathematica. – 1933. – №61. – P. 149-184.

39 Bohr H., Neugebauer O. Über lineare Differentialgleichung mit konstanten Koefficienten und fastperiodischen rechber Seite // Göttingen Nachrichten. – 1926. – vol. 1926. – P. 8-22.

40 Amerio L., Prouse G. Almost Periodic Functions and Functional Equations. – NY.: Van Nostrand Reinhold Company, 1961. – 184 p.

41 Харасахал А.Х. Почти-периодические решения обыкновенных дифференциальных уравнений. – Алма-Ата: Наука, 1970. – 200 с.

42 Жаутыков О.А. Распространение теорем Пуанкаре на бесконечные системы дифференциальных уравнений // В книге: Математическая физика. – Киев, 1966. – С. 3-22.

43 Zhautykov O.A. On the application of the shortening method to the problems of oscillations of elastic systems // Annuaire/Des ecoles superieures Mecanigue technique. – 1977. – Vol. l. – P. 7-14.

44 Умбетжанов Д.У. Почти многопериодические решения дифференциальных уравнений в частных производных. – Алма-Ата: Наука, 1979. – 204 с.

45 Умбетжанов Д.У. Почти периодические решения эволюционных систем. – Алма-Ата: Наука, 1990. – 184 с.

46 Умбетжанов Д.У., Бержанов А.Б. О голоморфном почти многопериодическом решении интегро-дифференциального уравнения в частных производных // Вестник Академии наук Казахской ССР, физико-математическая серия. – 1977. – №5. – С. 61-66.

47 Умбетжанов Д.У., Сартабанов Ж.А. О необходимом и достаточном условии многопериодичности решения одной системы дифференциальных уравнений в частных производных с одной и той же главной частью // В книге: Математика и Механика. – Алма-Ата, 1972. – № 7, часть 2. – С. 22-27.

48 Асанова А.Т. Почти периодическое решение решения полулинейного параболического уравнения // Дифференциальные уравнения. – 1998. – № 34:12. – С. 1697-1698.

49 Абдикаликова Г.А. Построение почти периодического решения одной квазилинейной параболической системы // Вестник Министерства образования и науки Республики Казахстан, физико-математическая серия. – 2002. – №3. – С. 3-7.

50 Sartabanov Z.A., Kulzhumiyeva A.A. On multiperiodic integrals of a linear system with the differentiation operator in the direction of the main diagonal in the space of independent variables // Eurasian Mathematical Journal. – 2017. – Vol. 8(1). – P. 67-75.

51 Oinarov R., Adiyeva A. Weighted inequality and oscillatory properties of a class of fourth order differential equations // Nonlinear Studies. – 2019. – Vol. 26(4). – P. 741-753.

52 Oinarov R., Rakhimova S.Y. Oscillation and nonoscillation of two terms linear and half-linear equations of higher order // Electronic Journal of Qualitative Theory of Differential Equations. – 2010. – №49. – P. 1-15.

53 Asanova A.T., Iskakova N.B., Orumbayeva N.T. Solvability of a periodic problem for the fourth order system of partial differential equations with time delay // Kazakh Mathematical Journal. – 2019. – Vol. 19, №2. – P. 14-21.

54 Raiko T., Valpola H. Oscillatory neural network for image segmentation with biased competition for attention // Advances in Experimental Medicine and Biology. – 2011. – №718. – P. 75-85.

55 Schmidt H., Avitabile D., Montbrió E., Roxin A. Network mechanisms underlying the role of oscillations in cognitive tasks // PLoS Computational Biology. – 2018. – Vol. 14(9). – P. e1006430.

56 Arbib M. The handbook of brain theory and neural networks. – Cambridge, MA: MIT Press, 2003. – 1309 p.

57 Koepsell K., Wang X., Hirsch J., Sommer F. Exploring the function of neural oscillations in early sensory systems // Frontiers in Neuroscience. – 2010. – №4. – P. 53-61.

58 Maguire M., Abel A. What changes in neural oscillations can reveal about developmental cognitive neuroscience: language development as a case in point // Developmental Cognitive Neuroscience. – 2013. – №6. – P. 125-136.

59 Ward L. Synchronous neural oscillations and cognitive processes // Trends in Cognitive Sciences. – 2003. – Vol. 7(12). – P. 553-559.

60 Akhmet M. Almost periodicity, chaos, and asymptotic equivalence. – Cham: Springer Nature Switzerland, 2020. – 368 p.

61 Akhmet M., Arugaslan D., Yilmaz E. Stability analysis of recurrent neural networks with piecewise constant argument of generalized type // Neural Networks. – 2010. – №23. – P. 305-311.

62 Akhmet M., Yilmaz E. Neural networks with discontinuous/impact activations. – NY.: Springer, 2013. – 176 p.

63 Cherif F. Existence and global exponential stability of pseudo almost periodic solution for SICNNs with mixed delays // Journal of Applied Mathematics and Computing. – 2012. – №39. – P. 235-251.

64 He Y., Wu M., She J. Delay-dependent exponential stability of delayed neural networks with time-varying delay // IEEE Transactions on Circuits and Systems - II Express Briefs. – 2006. – №53. – P. 553-557.

65 Li X., Song S. Impulsive control for existence, uniqueness, and global stability of periodic solutions of recurrent neural networks with discrete and continuously distributed delays // IEEE Transactions on Neural Networks and Learning Systems. – 2013. – Vol. 24(6). – P. 868-877.

66 Xiang H., Yan K.-M., Wang B.-M. Existence and global exponential stability of periodic solutions for delayed high-order Hopfield-type neural networks // Physics Letters A. – 2006. – Vol. 352(4-5). – P. 341-349.

67 Pan L., Cao J. Anti-periodic solutions for delayed cellular neural networks with impulsive effects// Nonlinear Analysis: Real World Applications. – 2011. – Vol. 12(6). – P. 3014-3027.

68 Wang L. Existence and global attractivity of almost periodic solutions for delayed high-order neural networks // Neurocomputing. – 2010. – №73. – P. 802-808.

69 Wen S., Huang T., Zeng Z., Chen Y., Li P. Circuit design and exponential stabilization of memristive neural networks // Neural Networks. – 2015. – №63. – P. 48-56.

70 Yu Y., Cai M. Existence and exponential stability of almost-periodic solutions for high-order Hopfield neural networks // Mathematical and Computer Modelling. – 2008. – Vol. 47(9-10). – P. 943-951.

71 Huang Z. Almost periodic solutions for fuzzy cellular neural networks with multi-proportional delays // International Journal of Machine Learning and Cybernetics. – 2017. – №8. – P. 1323-1331.

72 Das A., Roy A., Das P. Chaos in a three-dimensional neural network // Applied Mathematical Modelling. – 2000. – Vol. 24(7). – P. 511-522.

73 Choi T., Ahn C. Artificial life based on boids model and evolutionary chaotic neural networks for creating artworks // Swarm and Evolutionary Computation. – 2019. – №47. – P. 80-89.

74 Li Y., Oku M., He G., Aihara K. Elimination of spiral waves in a locally connected chaotic neural network by a dynamic phase space constraint // Neural Networks. – 2017. – №88. – P. 9-21.

75 Wan Y., Cao J., Wen G. Quantized synchronization of chaotic neural networks with scheduled output feedback control // IEEE Transactions on Neural Networks and Learning Systems. – 2016. – Vol. 28(11). – P. 2638-2647.

76 Hart J., Roy R., Muller-Bender D., Otto A., Radons G. Laminar chaos in experiments: nonlinear systems with time-varying delays and noise // Physical Review Letters. – 2019. – Vol. 123(15). – P. 154101.

77 Gray C., König P., Engel A., Singer W. Oscillatory responses in cat visual cortex exhibit inter-columnar synchronization which reflects global stimulus properties // Nature. – 1989. – Vol. 338(6213). – P. 334-337.

78 Korn H., Faure P. Is there chaos in the brain? II. Experimental evidence and related models // Comptes Rendus Biologies. – 2003. – №326. – P. 787-840.

79 Lassoued A., Boubaker O., Dhifaoui R., Jafari S. Experimental observations and circuit realization of a jerk chaotic system with piecewise nonlinear function // In book: Recent Advances in Chaotic Systems and Synchronization / ed. O. Boubaker, S. Jafari. – Cambridge: Elsevier, 2019. – P. 3-21.

80 Skarda C., Freeman W. How brains make chaos in order to make sense of the world? // Behavioral and Brain Sciences. – 1987. – №10. – P. 161-195.

81 Wanatabe M., Aihara K., Kondo S. Self-organization dynamics in chaotic neural networks // Control and Chaos. – 1997. – №8. – P. 320-333.

82 Birkhoff G. D. Dynamical Systems. – Providence, RI: American Mathematical Society. – 1927. – 319 p.

83 Miller A. Unpredictable points and stronger versions of Ruelle-Takens and Auslander-Yorke chaos // Topology and its Applications. – 2019. – №253. – P. 7-16.

84 Thakur R., Das R. Strongly Ruelle-Takens, strongly Auslander-Yorke and Poincaré chaos on semiflows // Communications in Nonlinear Science and Numerical Simulation. – 2020. – №81. – P. 1-6.

85 Bouzerdoum A., Pinter R. Shunting inhibitory cellular neural networks: derivation and stability analysis // IEEE Transactions on Circuits and Systems I Fundamental Theory and Applications. – 1993. – №40. – P. 215-221.

86 Hale J., Kocak H. Dynamics and bifurcations. – NY.: Springer-Verlag, 1991. – 574 p.

87 Hale J. Ordinary Differential Equations. – Malabar, FL, USA: Krieger Publishing Company, 1990. – 371 p.

88 Hartman P. Ordinary Differential Equations. – NY.: John Wiley, 1964. – 629 p.

89 Ou C. Almost periodic solutions for shunting inhibitory cellular neural networks // Nonlinear Analysis Real World Applications. – 2009. – №10. – P. 2652-2658.

90 Xu C, Li P. On anti-periodic solutions for neutral shunting inhibitory cellular neural networks with time-varying delays and D operator // Neurocomputing. – 2019. – №275. – P. 377-382.

91 Yu X., Wang Q. Weighted pseudo-almost periodic solutions for shunting inhibitory cellular neural networks on time scales // Bulletin of Malaysian mathematical science society. – 2019. – №42. – P. 2055-2074.

92 Li Y., Meng X. Almost periodic solutions for quaternion-valued shunting inhibitory cellular neural networks of neutral type with time delays in the leakage term // International Journal of Systems Science. – 2018. – №49. – P. 2490-2505.

93 Li Y., Wang L., Fei Y. Periodic solutions for shunting inhibitory cellular neural networks of neutral type with time-varying delays in the leakage term on time scales // Journal of Applied Mathematics. – 2014. – Vol. 2014. – P. 1-16.

94 Akhmet M., Fen M.O., Kivilcim A. Li-Yorke chaos generation by SICNNs with chaotic/almost periodic postsynaptic currents // Neurocomputing. – 2016. – №173. – P. 580-594.

95 Hopfield J.J. Neural networks and physical systems with emergent collective computational abilities // Proceedings of the National Academy of Sciences USA. – 1982. – №79. – P. 2554-2558.

96 Cohen M.A., Grossberg S. Absolute stability and global pattern formation and parallel memory storage by competitive neural networks // IEEE Transactions on System, Man, and Cybernetics. – 1983. – Vol. SMC-13, №7. – P. 815-821.

97 Mauro A., Conti F., Dodge F., Schor R. Subthreshold behavior and phenomenological impedance of the squid giant axon // The Journal of General Physiology. – 1970. – №55. – P. 497-523.

98 Koch C. Cable theory in neurons with active, linear zed membrane // Biological Cybernetics. – 1984. – №50. – P. 15-33.

99 Angelaki D.E., Correia M.J. Models of membrane resonance in pigeon semicircular canal type II hair cells // Biological Cybernetics. – 1991. – №65. – P. 1-10.

100 Juhong G., Jing X. Weak resonant double Hopf bifurcations in an inertial four-neuron model with time delay // International Journal of Neural Systems. – 2012. – Vol. 22(1). – P. 22-63.

101 Ke Y., Miao C. Stability analysis of BAM neural networks with inertial term and time delay // WSEAS Transactions on Systems and Control. – 2011. – Vol. 10(12). – P. 425-438.

102 Ke Y., Miao C. Stability and existence of periodic solutions in inertial BAM neural networks with time delay // Neural Computation and Applications. – 2013. – Vol. 23(3-4). – P. 1089-1099.

103 Qi J., Li C., Huang T. Stability of inertial BAM neural network with time-varying delay via impulsive control // Neurocomputing. – 2015. – №161. – P. 162-167.

104 Zhang W., Huang T., Li C., Yang J. Robust stability of inertial BAM neural networks with time delays and uncertainties via impulsive effect // Neural Processing Letters. – 2018. – Vol. 44(1). – P. 245-256.

105 Zhang Z., Quan Z. Global exponential stability via inequality technique for inertial BAM neural networks with time delay // Neurocomputing. – 2015. – №151. – P. 1316-1326.

106 Ke Y., Miao C. Stability analysis of inertial Cohen-Grossberg-type neural networks with time delays // Neurocomputing. – 2013. – №117. – P. 196-205.

107 Yu S., Zhang Z., Quan Z. New global exponential stability conditions for inertial Chen-Grossberg neural networks with time delays // Neurocomputing. – 2015. – №151. – P. 1446-1454.

108 Babcock K., Westervelt R. Stability and dynamics of simple electronic neural networks with added inertia // Physica D: Nonlinear Phenomena. – 1996. – Vol. 23(1). – P. 464-469.

109 Qin S., Gu L., Pan X. Exponential stability of periodic solution for a memristor-based inertial neural network with time delays // Neural Computation and Applications. – 2020. – №32. – P. 3265-3281.

110 Rakkiyappan R., Gayathri D., Velmurugan G., Cao J. Exponential synchronization of inertial memristor-based neural networks with time delay using average impulsive interval approach // Neural Processing Letters. – 2019. – Vol. 50(3). – P. 2053-2071.

111 Rakkiyappan R., Premalatha S., Chandrasekar A., Cao J. Stability and synchronization analysis of inertial memristive neural networks with time delays // Cognitive Neurodynamics. – 2016. – Vol. 10(5). – P. 437-451.

112 Driver R. Ordinary and delay differential equations. – NY.: Springer Science and Business Media, 2012. – 505 p.