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# DEVELOPMENT OF MATHEMATICAL AND NUMERICAL MODELS OF THE FILTRATION THEORY WITH CONSIDERING OF MASS TRANSFER PROCESSES

## ABSTRACT

**of the thesis for the degree of Doctor of Philosophy (PhD) in the specialty 8D05401 – Mathematics**

**General description of the thesis.** The work is devoted to the development of mathematical and numerical models of filtration theory taking into account mass transfer processes. In the course of the study, special attention is paid to the interaction between the liquid and the soil surface, as well as the effect of adsorption on the distribution and velocity of phase movement. This aspect plays a significant role in the accuracy of modeling and predicting multiphase filtration processes in real porous media.

**The actuality of the thesis.** In the modern oil industry, mathematical modeling is widely used in the design and development of oil fields. This makes it possible to solve the problems of monitoring and forecasting oil production and helps optimize its production and recovery of residual oil reserves. Extraction of residual oil reserves is the process of extracting oil from fields where long-term operational work has already been carried out, and the main production is nearing completion.

In addition to analytical methods for solving filtering problems, numerical methods are also used. In recent years, this field has gained a lot of experience in creating both mathematical and numerical models.

However, despite significant advances in the development of mathematical models of unsteady multiphase filtration, these achievements have not yet led to a radical increase in the efficiency of predicting the recovery of residual oil reserves, and are far from complete. The classical mathematical models of Musket-Leverett and Buckley-Leverett consider the unambiguous dependence of phase permeability on saturation and pressure. This is possible only in homogeneous formations. In practice, the oil reservoir is heterogeneous and also porous. Classical models are, in their own way, an equilibrium approximation. Nonequilibrium effects should also be taken into account in the model.

According to their rheological properties, the oils of Western Kazakhstan have a high viscosity.

Today, there is a problem of insufficient consideration of all nonequilibrium effects due to the complexity of physical processes (turbulence, chemical reactions, mass and heat transfer), limitations in computing resources, data availability and simplification of models.

All of the above factors indicate the relevance of the thesis, which examines the model of multiphase filtration, taking into account mass transfer processes. The model is a generalization of the Musket-Leverett model.

The research conducted in this dissertation is based on existing achievements in this field. It complements previous studies, taking into account not only the previously studied effects, but also adding a new aspect to the analysis - the influence of soil, taking into account adsorption. Thus, the scope of consideration is expanding, understanding and accounting for complex interactions in the system is deepening.

The thesis is based on the ideas of Barenblatt G.I., Konovalov A.N., Monakhov V.N., Mukhambetzhanov S.T.. The paper extends the analysis of difference schemes to mathematical models of oil displacement by water, taking into account thermal and chemical effects in the down hole zone of the reservoir. These research methods have promise and can be applied to similar problems related to related processes.

**The aim of the thesis research.** Development of mathematical and numerical models of filtration theory describing the motion of a multiphase liquid taking into account mass transfer processes and construction of approximate solutions to the problem with free boundaries.

**The research problems:**

a) to investigate a model of nonequilibrium isothermal filtration taking into account adsorption;

b) to investigate the behavior of the model at small values of the relaxation parameter;

c) to construct an approximate solution to the problem of nonequilibrium nonisothermal filtration based on the variational principle;

d) to investigate the qualitative properties of solving the problem of phase transitions in non-isothermal filtration with convective heat transfer;

e) to carry out a numerical study of the problem of non-isothermal filtration taking into account capillary forces;

f) to carry out a numerical study of the one-dimensional Stefan problem with movable boundaries in thermal conductivity and diffusion.

**Research methods.** Problems related to filtration theory are characterized by the use of modern functional methods to solve problems with moving boundaries in the filtration of multiphase media. The main research methods are classical methods of mathematical physics, difference methods, self-similar method, variational method and optimization method. Numerical analysis methods are also used.

**The object of research** is the problems of isothermal and non-isothermal nonequilibrium filtration.

**The scientific novelties:**

a) a mathematical model of nonequilibrium filtration with polymer solutions is substantiated. It is established that the system of Musket-Leverett equations, supplemented by the diffusion equation and kinetic relations, makes it possible to describe processes in formations when an active impurity is added; sufficient conditions for the existence and uniqueness of solutions are established, and the stability of solutions is proved. It is proved that the task is divided into three autonomous subtasks;

b) the conditions have been established under which the solution of the problem of nonequilibrium isothermal filtration tends to a generalized solution of the Stefan type problem;

c) the solution of the problem of non-isothermal filtration, taking into account mass transfer, is presented by an approximate solution method based on the variational principle;

d) sufficient conditions for the existence and uniqueness of the solution of the problem of phase transitions in non-isothermal filtration with convective heat transfer are indicated;

e) asymptotic estimates of the motion of the boundary of the phase transition are obtained;

e) the problem of non-isothermal filtration taking into account capillary forces using self-similar variables is reduced to a system of ordinary differential equations and a computational algorithm is proposed.;

f) an algorithm for numerically solving a problem with movable boundaries is presented, based on the finite element method, which makes it possible to track the position of phase boundaries with high accuracy, regardless of the number of phases or their changes over time.

**The results of the thesis which are taken out on defense:**

a) mathematical model of nonequilibrium isothermal and nonisothermal filtration;
b) convergence to a generalized solution of a Stefan type problem;
c) an approximate solution to the problem of non-isothermal filtration;
d) sufficient conditions for the existence and uniqueness of the solution of the problem of phase transitions in non-isothermal filtration with convective heat transfer;
e) asymptotic estimates of the motion of the boundary of the phase transition;
f) numerical studies based on capillary forces;
g) numerical solution algorithm with movable boundaries;
h) numerical study in self-similar variables.

**The validity and reliability** of the research is achieved by combining fundamental knowledge about physical processes with the use of mathematical models and numerical methods, as well as through an integrated approach that includes consideration of various aspects of physical processes, mathematical models and numerical methods.

The study is based on an analysis of the physical processes occurring in the oil production and filtration system through the soil. This includes consideration of phenomena such as heat transfer, mass transfer, adsorption in soil, paraffin formation, and others.

Mathematical models are used to describe physical processes, which are formulated based on the laws of physics, hydromechanics, and filtration theory. Variational methods, optimization methods, and the finite element method are used to analyze and solve mathematical models. These methods make it possible to obtain numerical solutions of equations approximating physical processes.

The reliability and validity is confirmed by publications in indexed international journals from the Scopus and Web of Science databases, as well as publications of the main results of activities in journals recommended by the Committee for Quality Assurance in Education and Science of the Ministry of Science and Higher Education of the Republic of Kazakhstan for the publication of the main results of scientific activities. In addition, the reliability has been confirmed by testing the results of the study in practice. The results of the dissertation were presented at seminars of the Department of Mathematics as they were received. at conferences. The reliability is justified by the given proofs of the theorems, the correctness of the transformations and confirmed by the results of computational experiments. The paper uses the results obtained by other authors earlier and marked with links. The reliability is also determined by the discussion of the results of the dissertation research at scientific seminars.

**Theoretical and practical significance of the research:**

a) the developed new mathematical model makes it possible to more accurately describe the processes of extraction of residual oil reserves;

b) the study of sufficient conditions of existence and uniqueness of solutions for problems of nonequilibrium filtration and phase transitions creates the basis for further research and applications in the framework of filtration theory;

c) establishing a connection between nonequilibrium nonisothermal filtration problems and generalized Stefan-type solutions makes it possible to deepen understanding of the dynamics of multiphase systems and their behavior under various conditions;

d) the obtained asymptotic estimates for the movement of the phase transition boundary can be used to optimize technological processes and improve the efficiency of field development.

**Approbation of the received results.** The results were presented at conferences of various levels. Below is a list where the results of the dissertation research were reported:

a) IX International Scientific Conference "Problems of differential equations, analysis and Algebra", Aktobe, 2022, report: "On joint motion of fluids in pore space";

b) International Mathematical Conference "Functional Analysis in interdisciplinary applications", 2023, Antalya, Turkey, report "Mathematical Modeling of Fluid Filtration Processes with Consideration of Mass Transfer Processes";

c) Proceedings of the International University Scientific Forum "Practice Oriented Science: UAE – RUSSIA – INDIA", 2023, report: "Estimation of fluid volume displacement in the reservoir using mathematical model";

d) XII International Scientific and Practical Conference "Innovative scientific research in the modern world: theory, methodology, practice", Ufa, 2023. report: "On the development of digital technology for oil and gas fields";

e) The traditional April International mathematical conference in honor of the Science Day of the Republic of Kazakhstan. Almaty, 2024, report: "Research of the mathematical model of nonequilibrium filtration";

f) Actual problems of applied mathematics and information technologies - Al-Khwarizmi 2024, report: Numerical Modeling of Fluid Filtration Processes with Free Boundaries.

The main results of the dissertation were presented at the scientific seminar of the Department of Mathematics of Aktobe Regional University named after K. Zhubanov "Problems of differential equations, applied mathematics and computer science" (scientific supervisor – Professor Zh.A. Sartabanov, secretary of the seminar – Associate Professor G.A. Abdikalikova) 12.01.2024, 02.21.2024, 02.27.2024, 10.10.2024, 11.7.2024, 18.02.2025.

**Articles.** 10 papers have been published on the topic of the dissertation, including 1 article in an international peer-reviewed scientific journal with a CiteScore index of 2.8 and a percentile in applied mathematics of 34 (Q3), 1 publication included in the WoS database (Q4), 2 publications in scientific journals included in the list recommended by the Committee for Quality Assurance in Science and Higher Education of the EOM of the Republic of Kazakhstan for the publication of the main results of scientific activity (list 1,2), 6 publications in the materials of international scientific conferences.

**The structure and scope of the dissertation.** The dissertation work consists of an introduction, two sections, the first section consists of 5 subsections and 5 paragraphs, the second section consists of 2 subsections and 3 paragraphs, a conclusion and a list of references.

**The number of illustrations, tables and references.** The work uses 8 illustrations, 4 tables and 94 sources.

**Summary of the dissertation.** In this paper, the main theoretical aspects of the study of the mathematical model of phase transitions in isothermal and non-isothermal filtration, as well as approximate methods used in solving problems of filtration theory taking into account mass transfer processes, are studied.

The introduction provides an overview of the literature on the topic of the dissertation research, summarizes the content of the work, the purpose, object, subject and objectives of the research, and formulates the results submitted for defense.

The first section of the paper contains five subsections. In the first subsection, the state of the issue of the joint movement of liquids in the pore space is highlighted, taking into account nonequilibrium effects. Concepts and formulas are given that are the basis for understanding the processes of joint movement of liquids in a porous space, taking into account nonequilibrium effects, and play an important role in the development of models and methods for analyzing transport in porous media.

In the second subsection, a mathematical model of isothermal filtration is considered. A mathematical model of nonequilibrium filtration, which describes the process of oil displacement by polymer solutions, is investigated. It is established that the system of Musket-Leverett equations, supplemented by the kinetic equation and the diffusion equation, makes it possible to describe the processes in oil reservoirs with the addition of an active impurity; sufficient conditions for the existence and uniqueness of solutions are established, and the stability of solutions is proved. It is proved that the task is divided into three autonomous subtasks. The main results are formulated in the form of theorems.

In the third subsection, the behavior of the initial system at small values of the relaxation parameter t and the behavior of the solution at the limiting transition $t \to 0$ are investigated. In the course of the study, it was concluded that the solution tends to a generalized solution of the Stefan problem. There are no moving boundaries in isothermal filtration, but with small relaxations, the isothermal system begins to behave like a non-isothermal system. At $t \to 0$, the isothermal system reacts quickly to changes and transitions between phases occur almost instantly. This is the key assumption of Stefan's problem. The conditions under which the solution of the problem of nonequilibrium isothermal filtration tends to a generalized solution of the Stefan type problem are established. As a result, the theorems are formulated.

In the fourth subsection, approximate methods for solving the problem of phase transitions in nonisothermal filtration are considered. In practice, an oil reservoir is a complex model in which nonequilibrium effects are present. Therefore, the transition to non-isothermal filtration was carried out and a model of non-isothermal filtration was built. The solution of the problem of non-isothermal filtration, taking into account mass transfer, is presented by an approximate solution method based on the variational principle. The functionals were formed and minimized (a measure of the error between the current and previous saturation values). The functionality was chosen in such a way as to describe the physical aspects of the filtration process. The main results are formulated in the form of theorems.

The fifth subsection examines the qualitative properties of solving the problem of phase transitions in nonisothermal filtration. The solvability of the Stefan problem with convective heat transfer is investigated in this section. Three phases are considered, water, oil and transition, as well as the dynamics of the phase transition. The position of the free boundary between the liquid and the solid phase, as well as the temperature distribution inside each phase, are described by self-similar functions, and the existence and uniqueness of a solution are proved. In the subsection, the asymptotic behavior of the solution of the problem is investigated, taking into account convection. Asymptotic estimates of the motion of the boundary of the phase transition are obtained.

Convection can accelerate or slow down the phase transition by changing the speed of the interface movement. The construction of this mathematical model is an urgent task for oil production. Temperature changes in formations, for example, during the intensification of oil production, can lead to phase changes in oil, such as crystallization. These processes can significantly affect the productivity of deposits. Modeling such processes is important for optimizing mining methods. Generalized solutions with a transition phase for the Stefan problem with convective heat transfer are found. In the above formulation, existence and uniqueness theorems are obtained, as well as qualitative properties of solutions are studied.

In the first subsection of the second section, the problem in self-similar variables is numerically investigated. Numerical calculations have shown that the water saturation and concentration of surfactants decrease with increasing spatial variable, which indicates an effective filtration process. The numerical methods used demonstrate high accuracy, which is confirmed by the stability of the results with an increase in the number of nodes in the grid. This indicates that the chosen numerical approach is correct.

A study was conducted on changes in water saturation and concentration in a porous medium during non-isothermal nonequilibrium filtration. As the filtration process progresses, the water saturation decreases, indicating a decrease in capillary pressure.

A numerical solution to the problem of capillary impregnation in a porous medium is presented, taking into account the processes of heat and mass transfer. The main attention is paid to the construction of a difference scheme and its implementation in conditions of non-isothermal filtration. The model used is described by a system of differential equations that includes the terms diffusion, filtration, and heat transfer. The main stages of the numerical algorithm included the discretization of a computational domain with a uniform grid in space and time, the use of the finite difference method to approximate derivatives, consideration of boundary conditions defining saturation at the edges of the domain, and an iterative process for calculating the saturation distribution based on stability criteria. Numerical experiments confirm the stability and convergence of the method. The behavior of the system shows the expected decrease in concentration in a porous medium and confirms the importance of taking capillary pressure and thermal conductivity into account in filtration tasks.  A computational algorithm in self-similar variables has been developed to solve the problem of nonequilibrium filtration. The main results are formulated as a theorem.