

ABSTRACT

of the dissertation work submitted for the degree of Doctor of Philosophy (PhD)
in the educational program 8D05401 – Mathematics

Tankeyeva Aigerim Kiyevna

Boundary value problems for quasilinear integro-differential equations with impulsive inputs: solution methods and properties

Structure and scope of the thesis. The dissertation consists of an introduction, two chapters (the first chapter includes four subsections and the second chapter includes three subsections), a conclusion and a list of references.

The number of illustrations, tables and references. The work contains 76 references, 2 tables and 10 figures.

The actuality of the dissertation. Integro-differential equations with impulsive inputs often arise in the modeling of physical systems that undergo abrupt changes at fixed moments of time. Therefore, it is important to develop new constructive methods that make it possible to effectively study boundary value problems for such equations and to find their solutions. These equations are widely applied in population biology, the propagation of chemical substances, heat transfer, electromagnetic wave radiation and other processes. Many researchers have studied them, including D.D. Bainov, P.S. Simeonov, A.M. Samoilenko, N.A. Perestyk, W.M. Haddad, V. Chellaboina, S.G. Neresov, A.O. Ignat'ev, J.J. Nieto, D. O'Regan, V. Lakshmikantham, J. Li and others. Despite this, the issues of solvability and construction of solutions for integro-differential equations with impulsive inputs remain a relevant problem. Therefore, it is important to develop new constructive methods that allow for the effective investigation of boundary value problems for such equations and the determination of their solutions.

The aim of the thesis research. The research objective is to study boundary value problems for Fredholm integro-differential equations with impulsive inputs using Dzhumabaev's parametrization method and on its basis, the construction of a constructive solution method.

The research problems:

- a) to establish the solvability of the special Cauchy problems for linear and quasilinear Fredholm integro-differential equations with impulsive inputs;
- b) to determine the general solution in the sense of Dzhumabaev for linear and quasilinear Fredholm integro-differential equations with impulsive inputs;
- c) to apply the general solution in the sense of Dzhumabaev to solve the boundary value problem for the linear Fredholm integro-differential equation with impulsive inputs;
- d) to establish the necessary and sufficient conditions for the solvability of the boundary value problem for the linear Fredholm integro-differential equation with impulsive inputs;

e) to develop an approximate numerical method and algorithms for solving the boundary value problem for the linear Fredholm integro-differential equation with impulsive inputs;

f) to determine the solvability of the special Cauchy problem for the system of quasilinear Fredholm integro-differential equations with impulsive inputs;

g) to find the general solution in the sense of Dzhumabaev for the quasilinear Fredholm integro-differential equation with impulsive inputs;

h) to reduce the boundary value problem for the quasilinear Fredholm integro-differential equation with impulsive effects to the system of quasilinear algebraic equations in the independent vectors of the general solution in the sense of Dzhumabaev.

The object of the research is boundary value problems for linear and quasilinear Fredholm integro-differential equations with impulsive inputs.

The subject of the research is the solvability problems of boundary value problems for linear and quasilinear Fredholm integro-differential equations with impulsive effects, the development of efficient algorithms for finding their solutions and their numerical analysis.

Scientific novelty.

1. The special Cauchy problem for the system of linear integro-differential equations with impulsive effects and parameters is solved.

2. The general solution in the sense of Dzhumabaev is constructed for a linear Fredholm integro-differential equation with impulsive inputs.

3. The boundary value problems for the linear Fredholm integro-differential equation with impulsive inputs were solved by applying the general solution in the sense of Dzhumabaev.

4. The necessary and sufficient conditions for the solvability of the boundary value problem for the linear Fredholm integro-differential equation with impulsive inputs have been established using the parametrization method.

5. An algorithm for finding the solution of the boundary value problem for the linear integro-differential equation with impulsive inputs has been developed and numerically implemented.

6. The special Cauchy problem for a system of quasilinear integro-differential equations with impulsive inputs and parameters has been solved.

7. The general solution in the sense of Dzhumabaev is constructed for the quasilinear Fredholm integro-differential equation with impulsive inputs.

8. The parameterization method has been extended to boundary value problems for a quasilinear Fredholm integro-differential equation with impulsive inputs.

The main results submitted for defense:

– the solvability of the special Cauchy problem for a system of linear Fredholm integro-differential equations with impulsive inputs;

– the $\Delta_N(\theta)$ general solution of the linear Fredholm integro-differential equation with impulsive inputs;

– the parametrization method for solving the boundary value problem for the linear Fredholm integro-differential equation with impulsive inputs;

– a numerical method and algorithms for finding an approximate solution to the boundary value problem for the linear Fredholm integro-differential equation with impulsive inputs;

– the solvability the special Cauchy problem for a system of a quasilinear Fredholm integro-differential equations with impulsive inputs;

– the general solution $\Delta_N(\theta)$ of a quasilinear Fredholm integro-differential equation with impulsive inputs;

– the construction of a system of quasilinear algebraic equations in the independent vectors of the $\Delta_N(\theta)$ general solution for the boundary value problem of the quasilinear Fredholm integro-differential equation with impulsive inputs.

Rationale for conducting the research. The dissertation widely employs the methods and results of the theory of differential, integro-differential and operator equations. The main method of analysis and solution of the considered problems is the parametrization method.

Practical and theoretical significance of the results obtained. The results of the dissertation are mainly theoretical in nature. The scientific significance of the work lies in the development of constructive methods for studying and solving boundary value problems for integro-differential equations with impulsive inputs.

The connection of the dissertation with other research works. The dissertation was carried out within the framework of the projects «Methods for solving boundary value problems for quasilinear impulsive systems of Fredholm integro-differential equations» (No. AP13268824, 2022-2024) and «New development of nonlinear operator equations and their applications» (No. AP23485509, 2024-2026) under the priority area «Fundamental research in the field of natural sciences» within the framework of grant funding.

The personal contribution of the author lies in the fact that all the results presented in the dissertation were obtained by the author independently. The contribution of the scientific supervisors consists in formulating the problems and discussing the obtained results.

Approbation of the research work. The main results of the work were presented and discussed at the following events:

– Traditional International Scientific April Conference. Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan (April 2023, April 2024);

– International Scientific Conference «Modern Problems of Differential Equations and Their Applications», Tashkent, November 23-25, 2023;

– Scientific seminar on Differential Equations at the National University of Uzbekistan (seminar leader – doctor of physical and mathematical sciences, professor Yuldashev T.K.);

– Scientific seminar of the Department of Differential Equations and Their Applications of the V.I. Romanovsky Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan (seminar leader – doctor of physical and mathematical sciences, associate professor Muminov Z.E.);

– Scientific seminar «Modern Problems of Differential and Integro-Differential Equations» of the Department of Differential Equations and Dynamical Systems of the Research Institute of Mathematics and Mathematical Modeling of the Ministry of

Science and Higher Education of the Republic of Kazakhstan (seminar leader – doctor of physical and mathematical sciences, professor Assanova A.T.);
 – Scientific seminar «Problems of Applied Mathematics and Informatics» of the Department of Mathematics of K. Zhubanov Aktobe Regional University (seminar leader – doctor of physical and mathematical sciences, professor Sartabanov Zh.A.).

An international research internship was conducted at the V.I. Romanovsky Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan from September 11, 2023 to October 20, 2023 (Tashkent, Uzbekistan).

Publications. The main results of the dissertation work were published in 8 scientific works, 2 articles in peer-reviewed scientific journals indexed in the Scopus and Web of Science databases, 1 article in a scientific publication included in List No. 1 of scientific journals recommended by the Committee for Quality Assurance in the Field of Science and Higher Education of the Ministry of Science and Higher Education of the Republic of Kazakhstan, 1 article in the scientific bulletin of the V.I. Romanovsky Institute of Mathematics of the Academy of Sciences of the Republic of Uzbekistan, 3 publications in the proceedings of international conferences and 1 certificate of registration in the State Register of Rights to Copyright-Protected Objects (object of copyright: computer software).

The main content of the dissertation. In the first chapter of the dissertation, boundary value problems for linear Fredholm integro-differential equations with impulsive inputs are studied. Subsection 1.1 examines the solvability of the special Cauchy problem for a system of linear Fredholm integro-differential equations with impulsive inputs

$$\frac{dx}{dt} = A(t)x + \sum_{k=1}^m \varphi_k(t) \int_0^T \psi_k(\tau)x(\tau)d\tau + f(t),$$

$$t \neq \theta_j, \quad j = \overline{1, l}, \quad t \in (0, T), \quad x \in R^n, \quad (1)$$

$$(\theta_0 = 0 < \theta_1 < \theta_2 < \dots < \theta_l < T = \theta_{l+1}),$$

$$B_j x(\theta_j - 0) + C_j x(\theta_j + 0) = d_j, \quad d_j \in R^n, \quad j = \overline{1, l}, \quad (2)$$

where the $n \times n$ matrices $A(t)$, $\varphi_k(t)$, $\psi_k(\tau)$, $k = \overline{1, m}$ are continuous on $[0, T]$, the n -vector function $f(t)$ is piecewise continuous on $[0, T]$ with possible continuous at the points $t = \theta_j$, $j = \overline{1, l}$, the $n \times n$ matrices B_j , C_j constant matrices and d_j a constant vector of dimension n .

A solution of problem (1), (2) is defined as a piecewise continuously differentiable function $x(t) = PC \left([0, T], R^n, \{\theta_j\}_{j=1}^l \right)$ on $(0, T)$ satisfying the integro-differential equation (1) and the impulsive input conditions (2).

The restriction of the function $x(t)$ to the r -th interval $[t_{r-1}, t_r)$ denote by $x_r(t)$, that is, $x_r(t) = x(t)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$. For definiteness, we will assume

that the values of the function $x_r(t)$ at the left endpoints of the interior intervals are equal to the right-hand limits, that is, $x_r(t_{r-1}) = \lim_{t \rightarrow t_{r-1}+0} x_r(t)$, $r = \overline{1, N}$.

Moreover, if the function $x(t)$ is piecewise continuously differentiable on the interval $(0, T)$ and satisfies the Fredholm integro-differential equation with impulsive inputs (1), (2) for each point $t \in (0, T) \setminus \{t_p, p = \overline{1, N-1}\}$ then the system of its restrictions $x[t] = (x_1(t), \dots, x_N(t))$ satisfies the following system of integro-differential equations

$$\frac{dx_r}{dt} = A(t)x_r + \sum_{k=1}^m \varphi_k(t) \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \psi_k(\tau) x_j(\tau) d\tau + f(t),$$

$$t \in [t_{r-1}, t_r), r = \overline{1, N}. \quad (3)$$

We introduce the parameters $\lambda_r = x_r(t_{r-1})$, $r = \overline{1, N}$ and $u_r(t) = x_r(t) - \lambda_r$ new unknown functions, if on each interval $[t_{r-1}, t_r)$ we substitute the expression for the function $x_r(t)$ we obtain the following system of integro-differential equations with parameters

$$\frac{du_r}{dt} = A(t)(u_r + \lambda_r) + \sum_{k=1}^m \varphi_k(t) \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \psi_k(\tau) (u_j(\tau) + \lambda_j) d\tau + f(t),$$

$$t \in [t_{r-1}, t_r), r = \overline{1, N}, \quad (4)$$

$$u_r(t_{r-1}) = 0, \quad r = \overline{1, N}. \quad (5)$$

In subsection 1.2, a new concept of the general solution in the sense of Dzhumabaev is introduced for linear Fredholm integro-differential equations with impulsive inputs (1), (2). The general solution plays an important role in the study and solution of various problems related to differential and integro-differential equations.

Definition 1. Let $u[t, \lambda] = (u_1(t, \lambda), \dots, u_N(t, \lambda))$ be the solution of the special Cauchy problem for the system of Fredholm integro-differential equations (4), (5). If its components satisfy the conditions

$$B_j \left(\lambda_{r+(j)} + \lim_{t \rightarrow \theta_j-0} u_{r+(j)}(t) \right) + C_j \lambda_{r+(j)+1} = d_j, \quad j = \overline{1, l},$$

then the function $x(\Delta_N(\theta), t, \lambda)$ defined by the equalities $x(\Delta_N(\theta), t, \lambda) = \lambda_r + u_r(t, \lambda)$, $t \in [t_{r-1}, t_r)$ and $x(\Delta_N(\theta), T, \lambda) = \lambda_N + \lim_{t \rightarrow T-0} u_N(t)$ is called the general solution $\Delta_N(\theta)$ in the sense of Dzhumabaev of the Fredholm integro-differential equation with impulsive effects (1), (2).

Thus, for each regular partition $\Delta_N(\theta)$ the function $x(\Delta_N(\theta), t, \lambda)$ is defined for all $\lambda = (\lambda_1, \dots, \lambda_N) \in R^{nN}$ and $t \in [0, T] \setminus \theta_j$, $j = \overline{1, l}$. The equivalence of the system of integro-differential equations (3) and the special Cauchy problem with a parameter (4), (5) means that for all $t \in (0, T) \setminus \{t_s, s = \overline{1, N-1}\}$ and $\lambda \in R^{nN}$ the function $x(\Delta_N(\theta), t, \lambda)$ satisfies the Fredholm integro-differential equation with impulsive inputs (1), (2)

$$x(\Delta_N(\theta), t, \lambda) = \lambda_r + \sum_{j=1}^N d_{r,j}(\Delta_N(\theta), t) \lambda_j + b_r(\Delta_N(\theta), t), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N}, \quad (6)$$

$$x(\Delta_N(\theta), T, \lambda) = \lambda_N + \sum_{j=1}^N d_{N,j}(\Delta_N(\theta), T) \lambda_j + b_N(\Delta_N(\theta), T). \quad (7)$$

Theorem 1. *If*

a) $\Delta_N(\theta)$ is a regular partition and the function $x(\Delta_N(\theta), t, \lambda)$ is the $\Delta_N(\theta)$ general solution of the Fredholm integro-differential equation with impulsive inputs (1), (2);

b) on the segment $[0, T]$ a piecewise continuous function $\tilde{x}(t)$ is given, with possible discontinuities at the points $t = t_r, r = \overline{1, N-1}$;

c) the function $\tilde{x}(t)$ has a piecewise continuous derivative on the interval $(0, T)$ with possible discontinuities at the points $\theta_j, j = \overline{1, l}$ and satisfies equation (1) under condition (2) for all $t \in (0, T) \setminus \{t_s, s = \overline{1, N-1}\}$;

then there exists a unique $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_N) \in R^{nN}$, such that for all $t \in [0, T]$ the equality $x(\Delta_N(\theta), t, \lambda) = \tilde{x}(t)$ holds.

Corollary 1. *If $\Delta_N(\theta)$ is a regular partition, $x(\Delta_N(\theta), t, \lambda)$ is the $\Delta_N(\theta)$ general solution of equations (1), (2) and function $\tilde{x}(t)$ is a solution of the Fredholm integro-differential equation (1) with condition (2), then there exists a vector $\tilde{\lambda} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_N) \in R^{nN}$, such that for all $t \in [0, T]$ the equality $x(\Delta_N(\theta), t, \lambda) = \tilde{x}(t)$ holds.*

In accordance with Corollary 1, for any solution of system (1), (2) there exists a unique parameter $\lambda = (\lambda_1, \dots, \lambda_N) \in R^{nN}$, thus, the general solution $\Delta_N(\theta)$ corresponds to the solution of system (1), (2). The solution of system (1), (2) compared to the piecewise continuously differentiable function $x(\Delta_N(\theta), t, \lambda)$, $t \in (0, T) \setminus \{t_p, p = \overline{1, N-1}\}$ is a piecewise continuous function on the segment $[0, T]$ differentiable on the interval $(0, T)$ with possible discontinuities at the points $\theta_j, j = \overline{1, l}$ and satisfies system (1), (2) for all $t \in (0, T)$. Therefore, if the function $x(t)$ is a solution of system (1), (2) then the system of functions $x[t] = (x_1(t), \dots, x_N(t))$ constructed from its restrictions, satisfies equation (1) with

the impulsive condition (2), as well as the continuity conditions at the points where no impulsive effect occurs

$$\lim_{t \rightarrow t_{r^-(p)} - 0} x_{r^-(p)}(t) - x_{r^-(p)+1}(t_{r^-(p)}) = 0, \quad p = \overline{1, N-l-1}. \quad (8)$$

Assume that $\Delta_N(\theta)$ is a regular partition and the function $x(\Delta_N(\theta), t, \lambda)$ is the $\Delta_N(\theta)$ general solution of system (1), (2). Using this solution, we write the impulsive conditions (2) and the continuity conditions (3) in the following form

$$B_j x(\Delta_N(\theta), \theta_j - 0, \lambda) + C_j x(\Delta_N(\theta), \theta_j + 0, \lambda) = d_j, \quad j = \overline{1, l}, \quad (9)$$

$$\lim_{t \rightarrow t_{r^-(p)} - 0} x(\Delta_N(\theta), t, \lambda) - x(\Delta_N(\theta), t_{r^-(p)}, \lambda) = 0, \quad p = \overline{1, N-l-1}. \quad (10)$$

Using the expression $x(\Delta_N(\theta), t, \lambda)$ of the general solution $\Delta_N(\theta)$ of the form (6), (7), and substituting it into conditions (9), (10), we obtain linear algebraic equations with $n(N-1)$ parameters

$$B_j [I + d_{r^+(j), r^+(j)}(\Delta_N(\theta), \theta_j)] \lambda_{r^+(j)} + [B_j d_{r^+(j), r^+(j)+1}(\Delta_N(\theta), \theta_j) + C_j] \lambda_{r^+(j)+1} + \sum_{\substack{i=1, i \neq r^+(j), \\ i \neq r^+(j)+1}}^N d_{r^+(j), i}(\Delta_N(\theta), \theta_j) \lambda_i = d_j - B_j b_{r^+(j)}(\Delta_N(\theta), \theta_j), \quad j = \overline{1, l}, \quad (11)$$

$$\begin{aligned} & [I + d_{r^-(p), r^-(p)}(\Delta_N(\theta), t_{r^-(p)})] \lambda_{r^-(p)} - [I - d_{r^-(p), r^-(p)+1}(\Delta_N(\theta), t_{r^-(p)}) + C_j] \times \\ & \times \lambda_{r^-(p)+1} + \sum_{\substack{i=1, i \neq r^-(p), \\ i \neq r^-(p)+1}}^N d_{r^-(p), i}(\Delta_N(\theta), t_{r^-(p)}) \lambda_i = \\ & = -b_{r^-(p)}(\Delta_N(\theta), t_{r^-(p)}), \quad p = \overline{1, N-l-1}. \end{aligned} \quad (12)$$

Lemma 1. *If the function system $x[t] = (x_1(t), \dots, x_N(t))$ is a solution of the system of integro-differential equations (3) and the functions $x_r(t)$, $r = \overline{1, N}$ satisfy conditions (6), then the function $x(t)$, defined by $x(t) = x_r(t)$, $t \in [t_{r-1}, t_r)$ $r = \overline{1, N}$ and $x(T) = \lim_{t \rightarrow T-0} x_N(t)$, is a solution of the Fredholm integro-differential equation with impulsive inputs (1), (2).*

Theorem 2. *For the system (1), (2) to be solvable, it is necessary and sufficient that the vector $b^*(\Delta_N(\theta))$ be orthogonal to the kernel of the transposed matrix $(D_*(\Delta_N(\theta)))'$, that is, the equality*

$$(b^*(\Delta_N(\theta)), \eta) = 0$$

holds for every $\eta \in \text{Ker} \left(D_*(\Delta_N(\theta)) \right)'$, where $(\cdot, \cdot)$ – denotes the inner product in the space $R^{n(N-1)}$.

Theorem 3. If $\Delta_N(\theta)$ is regular and the rank of the matrix $D_*(\Delta_N(\theta))$ is equal to $n(N-1)$, then the system (1), (2) has a classical general solution.

In subsection 1.3, the impulsive system (1), (2) with a boundary condition is considered

$$B_0 x(0) + C_0 x(T) = d_0, \quad d_0 \in R^n. \quad (13)$$

By substituting the corresponding expressions of the $\Delta_N(\theta)$ general solution into the boundary condition (13), we obtain a linear algebraic equation in parameters

$$B_0 \lambda_1 + C_0 \left[\lambda_N + \sum_{i=1}^N d_{N,i}(\Delta_N(\theta), T) \lambda_i \right] = d_0 - C_0 b_N(\Delta_N(\theta), T). \quad (14)$$

Let $Q_*(\Delta_N(\theta))$ denote the matrix corresponding to the left-hand side of equations (11), (12), (14). Then the system of equations (11), (12), (14) can be written in the following form

$$Q_*(\Delta_N(\theta)) \lambda = -F_*(\Delta_N(\theta)), \quad \lambda \in R^{nN}, \quad (15)$$

where $F_*(\Delta_N(\theta)) = (-d_1 + B_1 b_{r+(1)}(\Delta_N(\theta), \theta_1), \dots, -d_l + B_l b_{r+(l)}(\Delta_N(\theta), \theta_l),$

$$b_{r-(1)}(\Delta_N(\theta), t_{r-(1)}), \dots, b_{r-(N-l-1)}(\Delta_N(\theta), t_{r-(N-l-1)}),$$

$$-d_0 + C_0 b_N(\Delta_N(\theta), T)) \in R^{nN}.$$

Lemma 2. Suppose that $\Delta_N(\theta)$ is a regular partition and that the function $x(\Delta_N(\theta), t, \lambda)$ is the $\Delta_N(\theta)$ general solution of the Fredholm integro-differential system (1), (2) in the sense of Dzhumabaev.

(a) if $\lambda^* \in R^{nN}$ is a solution of equation (15), then the function $x^*(t) = x(\Delta_N(\theta), t, \lambda^*)$ is a solution of the boundary value problem (1), (2), (13);

(b) if $x^{**}(t)$ is a solution of the boundary value problem (1), (2), (13) and $\lambda_r^{**} = x^{**}(t_{r-1})$, $r = \overline{1, N}$ is a parameter composed of the components $\lambda^{**} = (\lambda_1^{**}, \dots, \lambda_N^{**}) \in R^{nN}$, then λ^{**} is a solution of equation (15).

Theorem 4. For the problem (1), (2), (13) to be solvable, it is necessary and sufficient that the vector $\mathcal{F}_*(\Delta_N(\theta))$ be orthogonal to the kernel of the transposed matrix $\left(Q_*(\Delta_N(\theta)) \right)'$, that is, the condition

$$(\mathcal{F}_*(\Delta_N(\theta)), \eta) = 0$$

holds for every $\eta \in \text{Ker} \left(Q(\Delta_N(\theta)) \right)'$, where $(\cdot, \cdot)$ – denotes the inner product in the space R^{nN} .

Corollary 2. For the problem (1), (2), (13) to have a unique solution, it is necessary and sufficient that the matrix $Q_*(\Delta_N(\theta)): R^{nN} \rightarrow R^{nN}$ be invertible.

In subsection 1.4, a numerical method and an algorithm for finding an approximate solution of boundary value problems for linear Fredholm integro-differential equations with impulsive effects are considered. The Dzhumabaev parameterization method is applied to boundary value problems for linear Fredholm integro-differential equations with impulsive effects (1), (2), (13). To demonstrate the implementation of the algorithm and the simplicity of its application, examples are provided.

In the second chapter of the dissertation, boundary value problems for quasilinear Fredholm integro-differential equations with impulsive inputs are studied. In subsection 2.1, to determine the solvability of the special Cauchy problem for a system of quasilinear Fredholm integro-differential equations with impulsive inputs, the quasilinear Fredholm integro-differential equation with impulsive effects at fixed moments of time is considered

$$\begin{aligned} \frac{dx}{dt} = & A(t)x + \sum_{k=1}^m \varphi_k(t) \int_0^T \psi_k(\tau)x(\tau)d\tau + f(t) + \\ & + \varepsilon f(t, x), \quad t \neq \theta_j, \quad j = \overline{1, l}, \quad t \in (0, T), \quad x \in R^n, \\ & (\theta_0 = 0 < \theta_1 < \theta_2 < \dots < \theta_l < T = \theta_{l+1}), \end{aligned} \quad (16)$$

$$B_j x(\theta_j - 0) + C_j x(\theta_j + 0) = d_j, \quad d_j \in R^n, \quad j = \overline{1, l}, \quad (17)$$

where $\varepsilon > 0$, $f: [0, T] \times R^n \rightarrow R^n$ continuous function, $\|x\| = \max_{i=\overline{1, n}} |x_i|$.

A function $x(t) \in PC \left([0, T], R^n, \{\theta_j\}_{j=1}^l \right)$ that is piecewise continuously differentiable on the interval $(0, T)$ and satisfies the integro-differential equation (16) and the impulsive conditions (17), is called a solution of the system (16), (17).

If the function $x(t)$ satisfies the Fredholm integro-differential equation with impulsive inputs (16), (17) and $(t, x(t)) \in G^{(0)}(\rho)$, then its system of restrictions $x[t] = (x_1(t), \dots, x_N(t))$ satisfies the following system of integro-differential equations

$$\begin{aligned} \frac{dx_r}{dt} = & A(t)x_r + \sum_{k=1}^m \varphi_k(t) \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \psi_k(\tau)x_j(\tau)d\tau + f_0(t) + \\ & + \varepsilon f(t, x_r), \quad t \in [t_{r-1}, t_r), \quad r = \overline{1, N}. \end{aligned} \quad (18)$$

By introducing additional parameters $\lambda_r \triangleq x_r(t_{r-1})$ and making the substitution $u_r(t) = x_r(t) - \lambda_r$ we obtain a system of integro-differential equations with parameters

$$\begin{aligned} \frac{du_r}{dt} = & A(t)(u_r + \lambda_r) + \sum_{k=1}^m \varphi_k(t) \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \psi_k(\tau)[u_j(\tau) + \lambda_j]d\tau + f_0(t) + \\ & + \varepsilon f(t, u_r + \lambda_r), \quad t \in [t_{r-1}, t_r), \end{aligned} \quad (19)$$

$$u_r(t_{r-1}) = 0, \quad r = \overline{1, N}. \quad (20)$$

The problem (19), (20) is the special Cauchy problem for a system of quasilinear Fredholm integro-differential equations with parameters.

Let us consider the linear special Cauchy problem corresponding to problem (19), (20)

$$\begin{aligned} \frac{du_r}{dt} = & A(t)(u_r + \lambda_r) + \sum_{k=1}^m \varphi_k(t) \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \psi_k(\tau)[u_j(\tau) + \lambda_j]d\tau + \\ & + f_0(t), \quad t \in [t_{r-1}, t_r), \end{aligned} \quad (21)$$

$$u_r(t_{r-1}) = 0, \quad r = \overline{1, N}. \quad (22)$$

Condition A. Let the following inequalities hold:

(1) $\|f(t, x)\| \leq M_0, (t, x) \in G^0(\rho), M_0 - \text{constant};$

(2) $M_1 \bar{h} = [(\alpha + K_0)(\rho + \|\lambda^{(0)}\|) + K_1 + \varepsilon M_0] \bar{h} < \rho, \text{ where}$

$$K_0 = \sum_{k=1}^m \max_{t \in [0, T]} \varphi_k(t) \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|\psi_k(\tau)\| d\tau, \quad \alpha = \max_{t \in [0, T]} A(t), \quad K_1 = \max_{t \in [0, T]} f_0(t),$$

$$\bar{h} = \max_{r=1, \overline{N}} (t_r - t_{r-1}).$$

let us choose the numbers $\rho_\lambda = \rho - M_1 \bar{h}$, $\rho_v = M_1 \bar{h}$ and construct the following sets:

$$S(\lambda^{(0)}, \rho_\lambda) = \{\lambda = (\lambda_1, \dots, \lambda_N) \in R^{nN}: \|\lambda_r - \lambda_r^{(0)}\| < \rho_\lambda, r = \overline{1, N}\},$$

$$S(0, \rho_v) = \{v[t] \in C([0, T], \Delta_N(\theta), R^{nN}): \|v[\cdot]\|_2 < \rho_v\},$$

$$G_p^0(\rho) = \{(t, x): t \in [t_{p-1}, t_p), \|x - x_0(t)\| < \rho - M_1(t_p - t)\}, p = \overline{1, N-1},$$

$$G_N^0(\rho) = \{(t, x): t \in [t_{N-1}, t_N], \|x - x_0(t)\| < \rho - M_1(t_N - t)\}$$

and

$$G_0(\Delta_N(\theta), \rho) = \bigcup_{r=1}^N G_r^0(\rho).$$

On the closed intervals, consider the following special Cauchy problem

$$\begin{aligned} \frac{dv_r}{dt} = & A(t)(v_r + \lambda_r) + \sum_{k=1}^m \varphi_k(t) \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \psi_k(\tau) [v_j(\tau) + \lambda_j] d\tau + f_0(t) + \\ & + \varepsilon f(t, v_r + \lambda_r), \quad t \in [t_{r-1}, t_r], \end{aligned} \quad (23)$$

$$v_r(t_{r-1}) = 0, \quad r = \overline{1, N}. \quad (24)$$

Theorem 5. Let the matrix $I - G(\Delta_N(\theta))$ be invertible, Condition A be satisfied and the following inequalities hold:

$$(i) \|f(t, x') - f(t, x'')\| \leq L_0 \|x' - x''\|, L_0 - \text{constant}, (t, x'), (t, x'') \in G_0(\rho);$$

$$(ii) (\alpha + L_0 + K_0) \bar{h} < 1;$$

$$(iii) \chi e^{\alpha(t_r - t_{r-1})} (\varepsilon M_0 + K_1 + (\alpha + K_0)(\rho_\lambda + \|\lambda^{(0)}\|)) \bar{h} < \rho_v, \text{ where}$$

$$\chi = 1 + \bar{h} \sum_{k=1}^m \max_{t \in [t_{r-1}, t_r]} \|\varphi_k(t)\| \sum_{p=1}^m \|\mathcal{R}_{k,p}(\Delta_N(\theta))\| \sum_{j=1}^N \int_{t_{j-1}}^{t_j} \|\psi_p(s)\| ds e^{\alpha \bar{h}}.$$

Then for any $\hat{\lambda} \in S(\lambda^{(0)}, \rho_\lambda)$ there exists a unique solution to the special Cauchy problem (23), (24) on the set $S(0, \rho_v)$, which is a system of functions $v[t, \hat{\lambda}] = (v_1(t, \hat{\lambda}), \dots, v_N(t, \hat{\lambda}))$.

Definition 2. If the system of functions $u[t, \lambda, \varepsilon] = (u_1(t, \lambda, \varepsilon), \dots, u_N(t, \lambda, \varepsilon)) \in S(v[t, \lambda], \rho_u)$ is the unique solution of the special Cauchy problem (19), (20) with parameters $\lambda = (\lambda_1, \dots, \lambda_N) \in S(\lambda^{(0)}, \rho_\lambda)$ and the following conditions are satisfied

$$B_j \left(\lambda_{r+(j)} + \lim_{t \rightarrow \theta_j - 0} u_{r+(j)}(t, \lambda, \varepsilon) \right) + C_j \lambda_{r+(j)+1} = d_j, \quad j = \overline{1, l},$$

then the function $x(\Delta_N(\theta), t, \lambda, \varepsilon)$ defined by the equalities $x(\Delta_N(\theta), t, \lambda, \varepsilon) = \lambda_r + u_r(t, \lambda, \varepsilon)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$ and $x(\Delta_N(\theta), T, \lambda, \varepsilon) = \lambda_N + \lim_{t \rightarrow T-0} u_N(t, \lambda, \varepsilon)$ is called the $\Delta_N(\theta)$ general solution in the sense of Dzhumabaev for the quasilinear integro-differential equation with impulsive inputs (16), (17) on the set $G^0(\Delta_N(\theta), \rho)$.

From Definition 2 and Theorem 5, the following result follows.

Theorem 6. If the conditions of Theorem 5 are satisfied, then the system (16), (17) has on the set $G^0(\Delta_N(\theta), \rho)$ a $\Delta_N(\theta)$ general solution in the form of the function $x(\Delta_N(\theta), t, \lambda, \varepsilon)$, which can be written as

$$x(\Delta_N(\theta), t, \lambda, \varepsilon) = y(\Delta_N(\theta), t, \lambda) + \Delta x(\Delta_N(\theta), t, \lambda, \varepsilon)$$

and the following estimate holds

$$\sup_{t \in [0, T]} \|\Delta x(\Delta_N(\theta), t, \lambda, \varepsilon)\| \leq \frac{1}{1 - q_\varepsilon} \varepsilon \chi \max_{r=\overline{1, N}} \sup_{t \in [t_{r-1}, t_r)} \|f(t, v_r(t, \lambda) + \lambda_r)\|,$$

where $\Delta x(\Delta_N(\theta), t, \lambda, \varepsilon) = u_r(t, \lambda, \varepsilon) - v_r(t, \lambda)$, $t \in [t_{r-1}, t_r)$, $r = \overline{1, N}$, $\Delta x(\Delta_N(\theta), T, \lambda, \varepsilon) = \lim_{t \rightarrow T-0} u_N(t, \lambda, \varepsilon) - \lim_{t \rightarrow T-0} v_N(t, \lambda)$.

In subsection 2.3 of the dissertation, boundary value problems for quasilinear Fredholm integro-differential equations with impulsive inputs are formulated. These problems are reduced to a system of quasilinear algebraic equations with respect to the independent vectors of the $\Delta_N(\theta)$ general solution. Consider, at fixed moments of time, the quasilinear Fredholm integro-differential equation with impulsive effects (16), (17) together with the boundary condition (3). For a regular partition $\Delta_N(\theta)$, by substituting the corresponding expressions of the general solution into the boundary condition (3), the impulsive conditions (17) and the continuity conditions (18), we obtain the following system of equations

$$B_0 x(\Delta_N(\theta), 0, \lambda) + C_0 x(\Delta_N(\theta), T, \lambda) = d_0, \quad (25)$$

$$B_j x(\Delta_N(\theta), \theta_j - 0, \lambda) + C_j x(\Delta_N(\theta), \theta_j + 0, \lambda) = d_j, \quad j = \overline{1, l}, \quad (26)$$

$$\lim_{t \rightarrow t_{r-(p)} - 0} x(\Delta_N(\theta), t, \lambda) - x(\Delta_N(\theta), t_{r-(p)}, \lambda) = 0, \quad p = \overline{1, N - l - 1}. \quad (27)$$

By substituting the $\Delta_N(\theta)$ general solution of the special Cauchy problem (19), (20) into the boundary condition (25), the impulsive conditions (26) and the continuity conditions (27), we obtain a system of quasilinear algebraic equations

$$Q_*(\Delta_N(\theta))\lambda = -F_*(\Delta_N(\theta)) - F^*(\Delta_N(\theta), \varepsilon, f). \quad (28)$$

Theorem 7. *Let the conditions of Theorem 5 and the following inequalities be satisfied:*

$$(i) Q_*(\Delta_N(\theta)) \text{ invertible and } \|[Q_*(\Delta_N(\theta))]\]^{-1} \leq \gamma;$$

$$(ii) q_\varepsilon = \varepsilon \chi L_0 < 1;$$

$$(iii) \sigma_\varepsilon = q_\varepsilon \left(\frac{\chi}{1 - q_\varepsilon} (\alpha + K_0 + \varepsilon L_0) + 1 \right) < 1;$$

$$(iv) \frac{1}{1 - q_\varepsilon} \varepsilon \chi \max_{r=1, N} \sup_{t \in [t_{r-1}, t_r]} \|f(t, v_r(t, \lambda^{(0)}) + \lambda_r^{(0)})\| < \rho_v;$$

$$(v) \frac{1}{1 - \sigma_\varepsilon} \frac{\varepsilon \chi}{1 - q_\varepsilon} \gamma \max(1, \|D\|) \max\left(\|d_0\| + \|C_0\|, \max_{j=1, l} (\|d_j\| + \|B_j\|), 1\right)$$

$$\times \max_{r=1, N} \sup_{t \in [t_{r-1}, t_r]} \|f(t, v_r(t, \lambda^{(0)}) + \lambda_r^{(0)})\| < \rho_\lambda.$$

Then the system of quasilinear algebraic equations (28) has a unique solution $\lambda = (\lambda_1, \dots, \lambda_N) \in S(\lambda^{(0)}, \rho_\lambda)$.

Main results of the dissertation:

– the solvability of the special Cauchy problem for a system of linear Fredholm integro-differential equations with impulsive inputs and parameters has been established;

– the general solution in the sense of Dzhumabaev has been constructed for the linear Fredholm integro-differential equation with impulsive inputs;

– the general solution in the sense of Dzhumabaev has been applied to solve the boundary value problem for the linear Fredholm integro-differential equation with impulsive inputs;

– the necessary and sufficient conditions for the solvability of the boundary value problem for the linear Fredholm integro-differential equation with impulsive inputs have been established;

– a numerical method and algorithms for finding an approximate solution to the boundary value problem for the linear Fredholm integro-differential equation with impulsive inputs have been proposed;

- the solvability of the special Cauchy problem for a system of quasilinear Fredholm integro-differential equations with impulsive inputs has been established;
- the general solution in the sense of Dzhumabaev has been constructed for the quasilinear Fredholm integro-differential equation with impulsive inputs;
- the boundary value problem for the quasilinear Fredholm integro-differential equation with impulsive effects has been reduced to a system of quasilinear algebraic equations with respect to the independent vectors of the general solution in the sense of Dzhumabaev.