ABSTRACT
of the dissertation work submitted for the degree of Doctor of Philosophy (PhD) in the educational program 8D05401 – Mathematics

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Mixed problems for degenerate multidimensional elliptic and elliptic-parabolic equations

Structure and volume of the dissertation. The dissertation consists of an introduction, three chapters (each containing 3 subsections), a conclusion, and a list of references.

Number of figures and references. The work includes 115 references and 2 figures.

The actuality of the dissertation. In the mathematical modeling of electromagnetic fields, it is known that the nature of the electromagnetic process depends on the properties of the medium. If the medium is in an equilibrium state, then according to Hamilton's principle we arrive at multidimensional elliptic equations.

If the medium has high conductivity, multidimensional parabolic equations arise. Consequently, the analysis of electromagnetic fields in a complex medium (for example, with varying conductivity) leads to the study of multidimensional elliptic–parabolic equations.

By methods of the theory of analytic functions of a complex variable, the well-posedness of boundary value problems for elliptic equations in the plane has been well studied. However, when the number of independent variables exceeds two, fundamental difficulties arise in studying similar problems. Due to the absence of a complete theory of multidimensional singular integral equations, the method of singular integral equations loses its effectiveness.

At the same time, the theory of multidimensional spherical functions is well developed. In generalized spaces, mixed problems for multidimensional hyperbolic equations have been studied in sufficient detail; in a number of works, the existence of a unique classical solution to these problems has been proven. Mixed problems for multidimensional elliptic equations have also been investigated.

However, mixed problems for degenerate elliptic and elliptic–parabolic equations have not yet been sufficiently studied. Therefore, the study of mixed problems for degenerate elliptic and elliptic–parabolic equations is an актуальной scientific problem.

The aim of the thesis research. to prove the uniqueness of solutions to mixed problems for degenerate multidimensional elliptic and elliptic–parabolic equations, as well as to identify new qualitative and structural properties of the solutions.

The research problems:

a) to prove the well-posedness of the mixed problem for a degenerate multidimensional elliptic equation, establish existence, and obtain an explicit analytical solution; determine conditions ensuring uniqueness;

b) to prove well-posedness for a class of degenerate multidimensional elliptic-parabolic equations, justify existence, construct explicit solutions, and prove uniqueness;

c) to study a generalized mixed problem for degenerate multidimensional elliptic-parabolic equations, prove well-posedness, existence and uniqueness, and derive explicit solutions.

The object of the research: Mixed problems for degenerate multidimensional elliptic and elliptic-parabolic equations in a cylindrical domain.

The subject of the research: problems of constructing solutions to mixed problems for degenerate multidimensional elliptic and elliptic–parabolic equations.

Scientific novelty

1. The existence of a solution to the mixed problem for degenerate multidimensional elliptic equations is proved, its analytical form is obtained, and well-posedness is established;

2. The uniqueness of the solution to the mixed problem for degenerate multidimensional elliptic equations is proved;

3. For a class of elliptic-parabolic equations, the well-posedness of the mixed problem is proved, solvability is established, and an explicit solution is obtained;

4. The uniqueness of the solution to the mixed problem for a class of elliptic-parabolic equations is proved;

5. The existence of a solution to the mixed problem for degenerate multidimensional elliptic-parabolic equations is proved, an explicit solution is obtained, and well-posedness is established;

6. The uniqueness of the solution to the mixed problem for degenerate multidimensional elliptic-parabolic equations is established.

Main results submitted for defense:

– the well-posedness of the mixed problem for degenerate multidimensional elliptic equations, the existence of its solution, its analytical representation, and the uniqueness of the solution;

– the well-posedness of the mixed problem for a class of degenerate multidimensional elliptic–parabolic equations, the existence of a solution, its explicit representation, and the uniqueness of the solution;

– the well-posedness of the mixed problem for degenerate multidimensional elliptic–parabolic equations, the existence of a solution, its explicit representation, and the uniqueness of the solution.

Theoretical and practical significance of the results obtained. The work is predominantly theoretical in nature. The obtained results can be used in the study of local and nonlocal boundary value problems for degenerate multidimensional elliptic and elliptic–parabolic equations.

The solutions obtained in the work can be applied in the analysis of numerical models of problems in gas dynamics, mechanics, biology, physics, economics, and other fields.

Author’s contribution. All the main results presented in the dissertation were obtained directly by the author. The formulation of the studied mixed problems, the selection of methods for their analysis, the justification of well-posedness, as well as the mathematical proofs of the existence and uniqueness of solutions were carried out independently by the author.

The author obtained analytical solutions to the problems using expansion methods in terms of spherical functions and Bessel functions.

The methods and approaches presented in the dissertation are a logical continuation and development of the results published by the author in scientific papers. These results have been tested at scientific conferences of various levels. In the course of the work, the author conducted an analysis of scientific literature and independently developed the theoretical provisions, proofs of theorems, and the dissertation text.

Approbation of the research work. The main results of the work were presented and discussed at the following events:

– IX International Scientific Conference “Problems of Differential Equations, Analysis and Algebra”. Aktobe Regional University named after K. Zhubanov. Aktobe, Kazakhstan (May 24–28, 2022);

– International scientific seminar “Problems of Differential Equations, Analysis and Algebra”, dedicated to the 70th anniversary of Doctor of Physical and Mathematical Sciences, Professor, Honorary Academician of the NAS RK K.K. Kenzhebaev. Aktobe Regional University named after K. Zhubanov. Aktobe, Kazakhstan (January 20, 2023);

– Traditional International Scientific Conference in April. Institute of Mathematics and Mathematical Modeling. Almaty, Kazakhstan (April 2024);

– Scientific seminar “Problems of Applied Mathematics and Informatics”, Aktobe Regional University named after K. Zhubanov, Department of Mathematics, Aktobe, Kazakhstan (seminar leader: Doctor of Physical and Mathematical Sciences, Professor Zh.A. Sartabanov), (2024–2026);

– International Scientific Conference “Fundamental and Applied Problems of Mathematics, Mechanics and Informatics”, dedicated to the 70th anniversary of Doctor of Physical and Mathematical Sciences, Professor Aibat Rafkhatovich Eshkeev. Karaganda, Kazakhstan (June 3–4, 2026).

The international internship took place from March 28 to April 26, 2022, in the Republic of Uzbekistan at Berdakh Karakalpak State University (Nukus).

Publications. A total of 9 works have been published on the topic of the dissertation, including 3 publications in ranked scientific journals indexed in the Scopus database, 2 articles in scientific publications included in the list recommended by the Committee for Quality Assurance in Science and Higher Education of the Ministry of Science and Higher Education of the Republic of Kazakhstan, 1 article in a scientific journal, as well as 3 articles in the proceedings of international conferences and seminars.

The main content of the dissertation. The dissertation consists of an introduction, three chapters, a conclusion, and a list of references. Each chapter, in turn, is divided into several subsections.

In the first chapter of the dissertation, mixed problems for degenerate multidimensional elliptic equations are considered.

1.1 Let us consider the formulation of a mixed problem for multidimensional elliptic equations.

Let D_α be a cylindrical domain of the Euclidean space E_{m+1} of points $(x_1, \dots, x_m, t)$, bounded by the cylinder $\Gamma = \{(x,t): |x| = 1\}$ and the planes плоскостями $t = \alpha > 0$ and $t = 0$, where $|x|$is the length of the vector $x = (x_1, \dots, x_m)$.

The parts of these surfaces forming the boundary ∂D_α of the domain D_α are denoted by $\Gamma_\alpha, S_\alpha, S_0$, respectively.

Definition. Let $D \subset R^n$ be an open domain, and let Γ be the boundary of the domain D. In the domain D, the functions $a_{ij}(x)$, $b_j(x)$, $c(x)$are given. Define the second-order differential operator L as follows:

$$Lu = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x)\frac{\partial u}{\partial x_j} + c(x)u.$$

If the quadratic form

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}(x)\ \xi_i \xi_j$$

is defined at every point of the domain D for any $\xi \in R^n$, then the differential operator L is called elliptic.

В области D_α рассмотрим взаимно-сопряженные вырождающихся многомерные эллиптические уравнения

$$Lu \equiv g(t)\Delta_x u + u_{tt} - \sum_{i=1}^{m} a_i(x,t)u_{x_i} + b(x,t)u_t + c(x,t)u = 0, \qquad (1)$$

$$L^* u \equiv g(t)\Delta_x v + v_{tt} - \sum_{i=1}^{m} a_i v_{x_i} + b v_t + c v = 0, \qquad (1*)$$

In the domain D_α, let us consider mutually adjoint degenerate multidimensional elliptic equations

$$Lu \equiv g(t)\Delta_x u + u_{tt} - \sum_{i=1}^{m} a_i(x,t)u_{x_i} + b(x,t)u_t + c(x,t)u = 0, \qquad (1)$$

$$L^* u \equiv g(t)\Delta_x v + v_{tt} - \sum_{i=1}^{m} a_i v_{x_i} + b v_t + c v = 0, \qquad (1*)$$

where $g(t)>0$ for $t>0$, $g(0)=0$, $g(t)\in C([0,\alpha])\cap C^2\big((0,\alpha)\big)$, Δ_x – is the Laplace operator with respect to
$x_1,\dots,x_m$, $m\geq 2$, a $d(x,t)=\mathrm{c}-\sum_{i=1}^{m}a_{ix_i}-b_t$.

The formulation of the mixed problem for degenerate multidimensional elliptic equations is as follows.

Problem 1. Find a solution of the equation

$$Lu\equiv g(t)\left(u_{rr}+\frac{m-1}{r}u_{rr}-\frac{\delta u}{r^2}\right)+u_{tt}+\sum_{i=1}^{m}a_i(r,\theta,t)u_{x_i}+b(r,\theta,t)u_t+ \qquad (1_1)$$

$$+c(r,\theta,t)u=0.$$

in the domain D_α from the class $C^1(\bar{D}_\alpha)\cap C^2(D_\alpha)$, satisfying the boundary conditions

$$u|_{S_0}=\tau(r,\theta),\quad u_t|_{S_0}=\nu(r,\theta),\quad u|_{\Gamma_\alpha}=\psi(t,\theta), \qquad (2)$$

with the compatibility conditions $\tau(1,\theta)=\psi(0,\theta)$, $\nu(1,\theta)=\psi_t(0,\theta)$.

Lemma 1. Let $f(r,\theta)\in W_2^l(S_0)$. If $l\geq m-1$, then the series

$$f(r,\theta)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}f_n^k(r)Y_{n,m}^k(\theta), \qquad (3)$$

as well as the series obtained from it by differentiation of order $p\leq l-m+1$, converge absolutely and uniformly.

$\{Y_{n,m}^k(\theta)\}$ -система линейно независимых сферических функций порядка n, $1\leq k\leq k_n$, $(m-2)!\,n!\,k_n=(n+m-3)!\,(2n+m-2),\theta=\big(\theta_{1,}\dots,\theta_{m-1}\big)$. $W_2^l(S_0), l=0,\dots,1$ - пространства Соболева.

$\{Y_{n,m}^k(\theta)\}$ – is a system of linearly independent spherical functions of order n, $1\leq k\leq k_n$, $(m-2)!\,n!\,k_n=(n+m-3)!\,(2n+m-2),\theta=\big(\theta_{1,}\dots,\theta_{m-1}\big)$. $W_2^l(S_0), l=0,\dots,1$ are Sobolev spaces.

Lemma 2. In order that $f(r,\theta)\in W_2^l(S_0)$, it is necessary and sufficient that the coefficients of series (3) satisfy the inequalities

$$|f_0^1(r)|\leq c_1,\quad \sum_{n=1}^{\infty}\sum_{k=1}^{k_n}n^{2l}\left|f_n^k(r)\right|^2\leq c_2,\quad c_1,c_2=\mathrm{const}$$

By $\tilde{a}_{in}^k(r,t)$, $a_{in}^k(r,t)$, $\tilde{b}_n^k(r,t),\tilde{c}_n^k(r,t),\hat{d}_n^k(r,t)$, $\rho_{\ n}^{\ k},\bar{\tau}_n^k(r)$, $\bar{\nu}_n^k(r)$, $\psi_n^k(t)$, we denote the coefficients of the series (1.3) corresponding to the functions

$$a_i(r,\theta,t)\rho(\theta),\qquad a_i\frac{x_i}{r}\rho,\quad b(r,\theta,t)\rho,\quad c(r,\theta,t)\rho,\quad d(r,\theta,t)\rho,\quad \rho(\theta),$$

$i = 1, \dots, m$, $\tau(r, \theta)$, $\nu(r, \theta)$, $\psi(t, \theta)$, where $\rho(\theta) \in C^{\infty}(H)$, H is the unit sphere in E_m.

1.2 Let us consider the well-posedness of the mixed problem for a degenerate multidimensional elliptic equation.

In the domain D_α, let us consider a degenerate multidimensional elliptic equation in the case when the additional terms in equation (1) are equal to zero:

$$g(t)\Delta_x u + u_{tt} = 0, \qquad (4)$$

where Δ_x is the Laplace operator.

Equation (4) in spherical coordinates takes the following form::

$$g(t)\left(u_{rr} + \frac{m-1}{r}u_r - \frac{1}{r^2}\delta u\right) + u_{tt} = 0,$$

Problem 1.1. In the domain D_α, it is required to find a solution of the elliptic equation from the class $C^1(\overline{D}_\alpha) \cap C^2(D_\alpha)$

$$g(t)\left(u_{rr} + \frac{m-1}{r}u_r - \frac{1}{r^2}\delta u\right) + u_{tt} = 0, \qquad (5)$$

satisfying the boundary conditions

$$u|_{S_0} = \tau(r, \theta), u_t|_{S_0} = \nu(r, \theta), u|_{\Gamma_\alpha} = \psi(t, \theta), \qquad (2)$$

where δ – is a second-order differential operator,

$$\delta \equiv -\sum_{j=1}^{m-1} \frac{1}{g_j sin^{m-j-1}\theta_j} \frac{\partial}{\partial\theta_j}\left(sin^{m-j-1}\theta_j \frac{\partial}{\partial\theta_j}\right)$$

$$g_1 = 1,\ g_j = \left(sin\theta_1 \dots sin\theta_{j-1}\right)^2, j > 1.$$

$$u(r, \theta, t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\left\{\psi_n^k(r) + r^{\frac{(1-m)}{2}}\left[\vartheta_{1n}^k(r, t) + \vartheta_{2n}^k(r, t)\right]\right\}Y_{n,m}^k(\theta), (6)$$

$$k_n \leq c_1 n^{m-2}, \left|\frac{\partial^l}{\partial\theta_j^l}Y_{n,m}^k(\theta)\right| \leq c_2 n^{\frac{m}{2}-1+l}, j = \overline{1, m-1}, l = 0,1, \dots . (7)$$

The spectrum of the operator δ consists of eigenvalues $\lambda_n = n(n + m - 2), n = 0,1, \dots$ each of which corresponds to k_n orthonormal functions $Y_{n,m}^k(\theta)$.

Theorem 1. If the conditions stated in Lemmas 1 and 2 are satisfied for the given functions in the boundary conditions (2), then the mixed problem 1.1 for equation (5) has a unique solution in the domain D_α, belonging to the class$C^1(\overline{D}_\alpha) \cap C^2(D_\alpha)$.

The solution is represented in the form (6) and satisfies estimate (7). Since this solution continuously depends on the functions specified in the boundary conditions, the stated mixed problem for the multidimensional elliptic equation is well-posed.

1.3 Existence and uniqueness of the solution of the mixed problem

In the domain D_α, let us consider degenerate multidimensional elliptic equations (1.1).

Let us consider Problem 1 as a mixed problem and show the existence of its solution.

Theorem 2. If $\tau(r,\theta),\ \nu(r,\theta) \in W_2^l(S_0), \psi(t,\theta) \in W_2^l(\Gamma_\alpha), l > \frac{3m}{2}$, then Problem 1 has a unique solution.

Let us construct a solution of the boundary value problem for equation (1*) with the data

$$v|_{\Gamma_\alpha} = 0, v|_{s_\alpha} = 0, v_t|_{s_\alpha} = \nu(r,\theta) = \bar{v}_n^k(r) Y_{n,m}^k(\theta), k = \overline{1, k_n}, n = 0,1,\ldots, (8)$$

where $\bar{v}_n^k(r) \in G, G$ is the set of functions $\nu(r)$ from the class $C([0,1]) \cap C^1([0,1])$. The set G is dense in $L_2\big((0,1)\big)$.

Thus, we arrive at the Dirichlet problem:

$$Lu = 0, \qquad u\Big|_{S_0} = 0, \quad u\Big|_{\Gamma_\alpha} = 0, \qquad u\Big|_{S_\alpha} = 0,$$

which has the zero solution.

Therefore, the uniqueness of the solution of the problem is established, and Theorem 2 is proved.

In the second chapter, a mixed problem for a class of degenerate multidimensional elliptic-parabolic equations is considered.

2.1 The formulation of a mixed problem for a class of elliptic-parabolic equations is presented.

Let $\Omega_{\alpha\beta}$ be a cylindrical domain of the Euclidean space E_{m+1} of points $(x_1, \ldots, x_m, t)$, bounded by the cylinder $\Gamma = \{(x,t): |x| = 1\}$ and the planes $t = \alpha > 0$ and $t = \beta > 0$ t=β>0, where $|x|$ is the length of the vector $x = (x_1, \ldots, x_m)$.

Let us denote by Ω_α and Ω_β the parts of the domain $\Omega_{\alpha\beta}$, and by $\Gamma_\alpha, \Gamma_\beta$ the parts of the surface Γ lying in the half-spaces $t > 0$ and $t > 0$; σ_α is the upper and σ_β is the lower base of the domain $\Omega_{\alpha\beta}$.

Let S be the common part of the boundary of the domains Ω_α and Ω_β, representing the set $\{t = 0, 0 < |x| < 1\}$ in E_m.

In the domain $\Omega_{\alpha\beta}$, consider degenerate multidimensional elliptic-parabolic equations

$$0=\begin{cases} t^q\Delta_x u-u_t+\sum_{i=1}^{m} d_i(x,t)u_{x_i}+e(x,t)u, t<0\\ |t|^q\Delta_x u-u_{tt}+\sum_{i=1}^{m} a_i(x,t)u_{x_i}+b(x,t)u_t+c(x,t)u, t<0, \end{cases} \quad (9)$$

where $p,q=const, p>0, q\geq 0, \Delta_x$ is the Laplace operator with respect to the variables $x_1,\dots,x_m, m\geq 2$.

In what follows, it is convenient to pass from Cartesian coordinates $x_1,\dots,x_m,t$ to spherical coordinates $r,\theta_1,\dots,\theta_{m-1},t, r\geq 0, 0\leq\theta_1<2\pi,\quad 0\leq\theta_i\leq\pi, i=2,3,\dots,m-1,\ \theta=\ \theta_1,\dots,\theta_{m-1}$.

As a mixed problem, let us consider the following problem.

Problem 2. Find a solution of equation (21) in the domain $\Omega_{\alpha\beta}$ for $t\neq 0$ from the class $C(\overline{\Omega}_{\alpha\beta})\cap C^1(\Omega_{\alpha\beta}\cup\Omega_\beta)\cap C^2(\Omega_\alpha\cup\Omega_\beta)$, satisfying the boundary conditions

$$L_1u\equiv t^q\left(u_{rr}+\frac{m-1}{r}u_r-\frac{1}{r^2}\delta u\right)-u_t+$$

$$+\sum_{i=1}^{m} d_i\,(r,\theta,t)u_{x_i}+e(r,\theta,t)u=0 \quad (2_1)$$

$$u\Big|_{\sigma_\alpha}=\varphi(r,\theta)\ ,\ u\Big|_{\Gamma_\alpha}=\psi_1(t,\theta), \quad (10)$$

$$u\Big|_{\Gamma_\beta}=\psi_2(r,\theta), \quad (11)$$

with the compatibility conditions $\varphi(1,\theta)=\psi_1(\beta,\theta),\psi_1(0,\theta)=\psi_2(0,\theta)$.

2.2 Well-posedness of the mixed problem and the structure of the solution for a class of elliptic-parabolic equations.

In this section, in the domain $\Omega_{\alpha\beta}$, we consider a degenerate multidimensional elliptic-parabolic equation

$$0=\begin{cases} t^q\Delta_x u-u_t, t>0\\ |t|^p\Delta_x u+u_{tt}, t<0. \end{cases} \quad (12)$$

Problem 2.1. In spherical coordinates, equation (12) in the domain Ω_α is written in the following form

$$t^q\left(u_{rr}+\frac{m-1}{r}u_r-\frac{1}{r^2}\delta u\right)-u_t=0 \quad (13)$$

It is required to find a solution belonging to the class $C(\overline{\Omega}_\alpha)\cap C^2(\Omega_\alpha)$ in the domain Ω_α, satisfying the conditions of the equation.

Since the desired solution of Problem 2 in the domain Ω_α belongs to the class $C(\bar{\Omega}_\alpha) \cap C^2(\Omega_\alpha)$, the solution of Problem 2.1 can be sought in the form

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} \bar{u}_n^k(r,t) Y_{n,m}^k(\theta), \tag{14}$$

where $\bar{u}_n^k(r,t)$ – are functions to be determined.

Thus, taking into account the boundary conditions in the domain Ω_β, we arrive at a mixed problem for a degenerate multidimensional elliptic equation

$$|t|^p \Delta_x u - u_{tt} = 0 \tag{15}$$

with the data

$$u\Big|_S = \tau(r,\theta), \quad u_t\Big|_S = \nu(r,0), u\Big|_{\Gamma_\beta} = \psi_2(t,\theta) \tag{16}$$

$$r^{-\frac{1}{2}} f_n^k(r,t) = \sum_{s=1}^{\infty} a_{s,n}(t) J_\nu(\mu_{s,n} r), \; r^{-\frac{1}{2}} \tilde{\varphi}_n^k(r) = \sum_{s=1}^{\infty} b_{s,n} J_\nu(\mu_{s,n} r), \tag{17}$$

Theorem 3. If the given functions in the boundary conditions (10), (11) satisfy the conditions of the lemmas and the convergence conditions, then the mixed problem (21) for equation (13) has a unique solution in the domain Ω_α.

This solution is determined by the expansion of the form (14) and is expressed in terms of Fourier–Bessel series (17), which, together with their derivatives, converge absolutely and uniformly. Therefore, the stated mixed problem is well-posed.

2.3 Existence and uniqueness of the solution of the mixed problem

In the domain $\Omega_{\alpha\beta}$, consider equations (7) and, as a mixed problem, consider Problem 2 and show its solvability.

Theorem 2. If $\varphi(r,\theta) \in W_2^p(S), \psi_1(t,\theta) \in W_2^p(\Gamma_\alpha), \psi_2(t,\theta) \in W_2^p(\Gamma_\beta),\; p > \frac{3m}{2}$ then Problem 2 is uniquely solvable.

First, let us show the existence of a solution to problem (9), (10) in the domain Ω_α. Consider equation (21) in spherical coordinates.

The desired solution of Problem 2 in the domain Ω_α will be sought in the following form

$$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} \bar{u}_n^k(r,t) Y_{n,m}^k(\theta), \tag{18}$$

where $\bar{u}_n^k(r,t)$ are functions to be determined.

$$u(r,\theta,0)=\tau(r,\theta)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\tau_n^k(r)Y_{n,m}^k(\theta),$$

$$\tau_n^k(r)=\psi_{1n}^k(0)+\sum_{s=1}^{\infty}r^{\frac{(2-m)}{2}}\left[\int_0^{\alpha}a_{s,n}(\xi)\exp\left(\frac{\mu_{s,n}^2}{q+1}\xi^{q+1}\right)d\xi+\right.$$
$$\left.+b_{s,n}\left(\exp\frac{\mu_{s,m}^2}{q+1}\alpha^{q+1}\right)\right]J_{n+\frac{(m-2)}{2}}\left(\mu_{s,n}r\right). \tag{19}$$

$$u_t(r,\theta,0)=\nu(r,\theta)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\nu_n^k(r)Y_{n,m}^k(\theta),$$

$$\nu_n^k(r)=\psi_{1nt}^k(0)-\sum_{s=1}^{\infty}r^{\frac{2-m}{2}}a_{s,n}(0)J_{n+\frac{(m-2)}{2}}\left(\mu_{s,n}r\right). \tag{20}$$

$$u(r,\theta,0)=\tau(r,\theta)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\tau_n^k(r)Y_{n,m}^k(\theta),$$

$$\tau_n^k(r)=\psi_{1n}^k(0)+\sum_{s=1}^{\infty}r^{\frac{(2-m)}{2}}\left[\int_0^{\alpha}a_{s,n}(\xi)\exp\left(\frac{\mu_{s,n}^2}{q+1}\xi^{q+1}\right)d\xi+\right.$$
$$\left.+b_{s,n}\left(\exp\frac{\mu_{s,m}^2}{q+1}\alpha^{q+1}\right)\right]J_{n+\frac{(m-2)}{2}}\left(\mu_{s,n}r\right). \tag{21}$$

$$u_t(r,\theta,0)=\nu(r,\theta)=\sum_{n=0}^{\infty}\sum_{k=1}^{k_n}\nu_n^k(r)Y_{n,m}^k(\theta),$$

$$\nu_n^k(r)=\psi_{1nt}^k(0)-\sum_{s=1}^{\infty}r^{\frac{2-m}{2}}a_{s,n}(0)J_{n+\frac{(m-2)}{2}}\left(\mu_{s,n}r\right). \tag{22}$$

Thus, taking into account the boundary conditions (11), (21), (22) in the domain Ω_α, we arrive at a mixed problem for degenerate multidimensional elliptic equations

$$L_2u\equiv|t|^{\rho}\Delta_x u+u_{tt}+\sum_{i=1}^{m}a_i(r,\theta,t)u_{x_i}+b(r,\theta,t)u_t+c(r,\theta,t)u=0, \tag{23}$$

with the data

$$u\Big|_S = \tau(r,\theta), \quad u_t\Big|_S = \nu_2(r,\theta), \quad u\Big|_{\Gamma_\beta} = \psi_2(t,\theta). \tag{24}$$

First, let us consider problem (7), (8) in the domain Ω_d and prove the uniqueness of its solution. For this purpose, we construct a solution of the first boundary value problem for the equation

$$L_1^* v \equiv t^q \Delta_x v - v_t - \sum_{i=1}^{m} d_i v_{x_i} + dv = 0, \tag{2_1^*}$$

with the data

$$v\Big|_S = \tau(r,\theta) = \bar{\tau}_n^k(r) Y_{n,m}^k(\theta), \quad v\Big|_{\Gamma_\alpha} = 0, \tag{25}$$

where $d(x,t) = e - \sum_{i=1}^{m} d_i x_i$, $\bar{\tau}_n^k(r) \in G,$ *G is the set of functions* $\tau(r)$ from the class $C([0,1]) \cap C^1\big((0,1)\big)$, the set G is dense in $L_2\big((0,1)\big)$.

$$L_1 \vartheta_n^k = t^q \left(\vartheta_{nrr}^k + \frac{\bar{\lambda}_n}{r^2} \vartheta_n^k \right) - \vartheta_{nt}^k = \tilde{f}_n^k(r,t), \tag{26}$$

Thus, by the extremum principle for parabolic equations (2_1) ([23]) $u \equiv 0$ in $\bar{\Omega}_\alpha$. It follows that

$$u_t(r,\theta,0) = v(r,\theta) = 0, \forall (r,\theta) \in S.$$

Thus, we arrive at the homogeneous mixed problem (25), (26), which, by virtue of Theorem 1, has the trivial solution.

In the third chapter, a mixed problem for degenerate multidimensional elliptic-parabolic equations is considered.

3.1 The formulation of a mixed problem for elliptic-parabolic equations is presented.

Let $\Omega_{\alpha\beta}$ be a cylindrical domain of the Euclidean space E_{m+1} of points $(x_1,\ldots,x_m,t)$ bounded by the cylinder $\Gamma = \{(x,t);\ |x| = 1\}$, and the planes $t = \alpha > 0$ and $t = \beta < 0$ where $|x|$ is the length of the vector $x = (x_1,\ldots,x_m)$.

Let us denote by Ω_α and Ω_β the parts of the domain $\Omega_{\alpha\beta}$, and by $\Gamma_\alpha, \Gamma_\beta$ the parts of the surface Γ, lying in the half-spaces $t > 0$ and $t < 0$, σ_α is the upper and σ_β is the lower base of the domain $\Omega_{\alpha\beta}$.

Let S - be the common part of the boundaries of the domains Ω_α and Ω_β, representing the set $\{t = 0, 0 < |x| < 1\}$ in E_m.

In the domain $\Omega_{\alpha\beta}$ consider degenerate elliptic-parabolic equations

$$0=\begin{cases} g(t)\Delta_x u-u_t+\sum_{i=1}^{m} d_i\,(x,t)u_{x_i}+e(x,t)u, t>0, \\ p(t)\Delta_x u+u_{tt}+\sum_{i=1}^{m} a_i(x,t)u_{x_i}+b(x,t)u_t+c(x,t)u, t<0, \end{cases} \quad (27)$$

where g(t) >0 for $t>0, g(0)=0, g(t)\in C([0,\alpha])\cap C^2\big((0,\alpha)\big)$,
p(t) >0 for $\mathrm{t}<0, \mathrm{p}(0)=0, \mathrm{p(t)}\in \mathrm{C}([\beta,0])$,
Δ_x is the Laplace operator with respect to the variables $x_1,\dots,x_m, m\geq 2$.

Problem 3. Find a solution of equation (27) in the domain $\Omega_{\alpha\beta}$ for $\mathrm{t}\neq 0$ from the class $\mathrm{C}\big(\overline{\Omega}_{\alpha\beta}\big)\cap \mathrm{C}^1\big(\Omega_{\alpha\beta}\big)\cap \mathrm{C}^2\big(\Omega_\alpha\cup\Omega_\beta\big)$, satisfying the boundary conditions

$$u|_{\Gamma_\alpha}=\psi_1(t,\theta), u|_{\sigma_\alpha}=\varphi(r,\theta), \quad (28)$$

$$u|_{\Gamma_\beta}=\psi_2(t,\theta), \quad (29)$$

$$L_1u=g(t)\left(u_{rr}+\frac{m-1}{r}u_r-\frac{1}{r^2}\delta u\right)-u_t+ \quad (3_1)$$
$$+\sum_{i=1}^{m} d_i(r,\theta,t)u_{x_i}+e(r,\theta,t)u=0,$$

$$\delta\equiv-\sum_{j=1}^{m-1}\frac{1}{g_j sin^{m-j-1}\theta_j}\frac{\partial}{\partial\theta_j}\left(sin^{m-j-1}\theta_j\frac{\partial}{\partial\theta_j}\right)$$

$$g_1=1,\ g_j=\left(sin\theta_1\dots sin\theta_{j-1}\right)^2, j>1.$$

3.2 Well-posedness of the mixed problem

In this section, in the domain $\Omega_{\alpha\beta}$, we consider a degenerate multidimensional elliptic-parabolic equation

$$0=\begin{cases} g(t)\Delta_x u-u_t, t>0 \\ p(t)\Delta_x u+u_{tt}, t<0 \end{cases} \quad (30)$$

In spherical coordinates, equation (30) in the domain Ω_α takes the following form.

Problem 3.1. In the domain Ω_α, it is required to find a solution of the equation from the class$C(\overline{\Omega}_\alpha)\cap C^2(\Omega_\alpha)$

$$g(t)\left(u_{rr}+\frac{m-1}{r}u_r-\frac{1}{r^2}\delta u\right)-u_t=0. \quad (31)$$

satisfying the boundary conditions (28), (29).

Since the desired solution of Problem 3 in the domain Ω_α belongs to the class$C(\bar{\Omega}_\alpha) \cap C^2(\Omega_\alpha)$, it can be sought in the form

$$u(r,\theta,t) = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} \bar{u}_n^k\,(r,t) Y_{n,m}^k(\theta), \tag{32}$$

where $\bar{u}_n^k(r,t)$are functions to be determined.

$$T_{s,n}(t) = \left(exp\left(-\mu_{s,n}^2 \int_0^t g(\xi) d\xi \right) \right) \times$$
$$\times \left(\int_t^\alpha a_{s,n}(\xi) \left(exp\,\mu_{s,n}^2 \int_0^\xi g(\xi_1) d\xi_1 \right) d\xi \right). \tag{33}$$

$$\tau_n^k(r) = \psi_{1n}^k(0) + \sum_{s=1}^{\infty} r^{\frac{2-m}{2}} \left[\int_0^\alpha a_{s,n}(\xi) \left(\exp\left(\mu_{s,n}^2 \int_0^\xi g(\xi_1) \xi_1 \right) \right) d\xi + \right.$$
$$\left. + b_{s,n} \exp\left(\mu_{s,n}^2 \int_0^\alpha g(\xi) d\xi \right) J_{n+\frac{m-2}{2}}(\mu_{s,n} r). \right. \tag{34}$$

$$a_{s,n}(t) = 2\left[J_{\nu+1}(\mu_{s,n})\right]^{-2} \int_0^1 \sqrt{\xi}\, f_n^k(\xi,t) J_\nu(\mu_{s,n}\xi) d\xi, \tag{35}$$

$$b_{s,n} = 2\left[J_{\nu+1}(\mu_{s,n})\right]^{-2} \int_0^1 \sqrt{\xi}\, \tilde{\varphi}_{1n}^k(\xi) J_\nu(\mu_{s,n}\xi) d\xi, \tag{36}$$

$$u_t(r,\theta,0) = \nu(r,\theta) = \sum_{n=0}^{\infty} \sum_{k=1}^{k_n} \nu_n^k(r) Y_{n,m}^k(\theta), \tag{37}$$

Thus, taking into account the boundary conditions (29), (34), (36) in the domain Ω_α, we arrive at a mixed problem for degenerate multidimensional elliptic equations

$$g(t)\Delta_x u + u_{tt} = 0 \tag{38}$$

with the data

$$u|_S = \tau(r,\theta), \quad u|_S = \upsilon_2(r,\theta), \quad u|_{\Gamma_\alpha} = \psi_2(t,\theta).$$

In Section 1.2, it was proved that this problem has a unique solution (Theorem 1).

Theorem 5. If the boundary conditions (29) are satisfied and the coefficients (33), (35), (34) satisfy the convergence conditions of the Fourier–Bessel expansion, then the mixed problem (31) for equation (3_1) has a unique solution in the domain Ω_α, belonging to the class $C(\overline{\Omega}_\alpha) \cap C^2(\Omega_\alpha)$.

Moreover, since problem (3_1) can be reduced to problem (1_1)–(2) considered in Section 1.2, its solution exists, is unique, and depends continuously on the given data. Therefore, problem (3_1)–(31) is well-posed.

3.3 Solvability and uniqueness of the solution of the mixed problem.

In this section, in the domain $\Omega_{\alpha\beta}$ we consider equation (3_1) and, as a mixed problem, consider Problem 3 and show its solvability.

Theorem 6. If $\varphi(r,\theta) \in W_2^p(S), \psi_1(t,\theta) \in W_2^p(\Gamma_\alpha)\psi_2(t,\theta) \in W_2^2(\Gamma_\beta)$, $p > \frac{3m}{2}$, then Problem 3 has a unique solution.

First, let us show the solvability of problem (27), (28). Consider equation (27) in spherical coordinates in the domain Ω_α, i.e., we obtain equation (3_1).

$$u(r,\theta,0) = \tau(r,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} \tau_n^k(r) Y_{n,m}^k(\theta), \tag{39}$$

$$\tau_n^k(r) = \psi_{1n}^k(0) + \sum_{s=1}^{\infty} r^{\frac{(2-m)}{2}} \left[\int_0^\alpha a_{s,n}(\xi) \left(exp\, \mu_{s,n}^2 \int_0^\xi g(\xi_1) d\xi_1 \right) d\xi + \right.$$

$$\left. + b_{s,n}\, exp \left(\mu_{s,n}^2 \int_0^\alpha g(\xi) d\xi \right) \right] J_{n+(m-2)/2}(\mu_{s,n} r),$$

$$u_t(r,\theta,0) = \nu(r,\theta) = \sum_{n=0}^{\infty}\sum_{k=1}^{k_n} \nu_n^k(r) Y_{n,m}^k(\theta), \tag{40}$$

Thus, taking into account the boundary conditions in the domain Ω_α, we arrive at a mixed problem for degenerate multidimensional elliptic equations

$$L_2 u = p(t)\Delta_x u + u_{tt} + \textstyle\sum_{l=1}^m a_i(r,\theta,t) u_{x_i} + b(r,\theta,t) u_t + c(r,\theta,t) u = 0, \tag{41}$$

with the data

$$u\Big|_S = \tau(r,\theta),\ u_t\Big|_S = \nu(r,\theta),\ u\Big|_{\Gamma_\beta} = \psi_2(t,\theta) \qquad (42)$$

According to Theorem 2 proved in the first chapter, if $\tau(r,\theta), \nu(r,\theta) \in W_2^l(S)$, $\psi_2 \in W_2^l(\Gamma_\beta)$, $l > \frac{3m}{2}$, then problem (41), (42) has a unique solution.

Further, using this theorem, we obtain the solvability of Problem 3.

Thus, by the extremum principle for the parabolic equation (3_1) $u \equiv 0$ in $\overline{\Omega}_\alpha$.

It follows that $u_t(r,\theta,0) = \nu(r,\theta,0) = 0, \forall (r,\theta) \in S$.

Thus, we arrive at the homogeneous mixed problem (41), (42), which, by virtue of Theorem 1, has the trivial solution. Therefore, the uniqueness of the solution of Problem 3 is proved.

Main results of the dissertation:

– a mixed problem for degenerate multidimensional elliptic equations is formulated, its solution is sought in the form of a series in spherical coordinates, and the existence of the solution and the well-posedness of the problem are proved;

– the uniqueness of the solution to the mixed problem for degenerate multidimensional elliptic equations is proved;

– for a class of elliptic-parabolic equations, a mixed problem is formulated, its well-posedness is established, the existence of the solution is proved, and an explicit solution of this problem is obtained;

– the uniqueness of the solution to the mixed problem for a class of elliptic-parabolic equations is proved;

– the existence of the solution to the mixed problem for degenerate multidimensional elliptic-parabolic equations is proved, its explicit solution is obtained, and well-posedness is established;

– the uniqueness of the solution to the mixed problem for degenerate multidimensional elliptic-parabolic equations is established.