ABSTRACT

of the dissertation work submitted for the degree of Doctor of Philosophy (PhD) in the educational program 8D05401 – Mathematics

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Approximate solutions of the Riemann problem for two-phase flow of immiscible liquids based on the Buckley-Leverett model

Structure and scope of the dissertation. The dissertation consists of an introduction, two sections (the first section contains 4 subsections, the second section contains 7 subsections), a conclusion, a list of references, and appendices.

Number of figures, tables, and references. The work includes 120 references, 2 tables, and 28 figures.

The numbering of formulas, definitions, and theorems consists of two numbers: the first denotes the section number, and the second denotes the corresponding number of the formula or theorem within the section. The dissertation comprises 111 pages.

Relevance of the research topic. Mathematical models describing the flow of two-phase immiscible fluids in porous media currently occupy an important place in petroleum engineering, hydrogeology, and problems of efficient use of energy resources. One of the phenomena underlying such processes is the displacement of one phase by another, resulting in the formation of shock waves, rarefaction regions, and frontal movement.

One of the classical models describing these phenomena is the Buckley – Leverett equation. This model belongs to hyperbolic conservation laws, and its solutions generally have a discontinuous character. In this regard, the correct definition of weak and entropy solutions, as well as ensuring their uniqueness and stability, is an important issue from both theoretical and applied perspectives.

In recent years, the vanishing viscosity method has been widely used as one of the most reliable approaches for selecting entropy solutions of hyperbolic conservation laws. This method approximates a hyperbolic equation by a parabolic equation with small viscosity and makes it possible to select the limiting solution, as the viscosity parameter tends to zero, as the entropy solution.

From this point of view, the study of approximate solutions of the Riemann problem for the Buckley – Leverett model obtained by the vanishing viscosity method, the analysis of the structure of the shock layer, and their justification using numerical models constitute a relevant and significant scientific problem.

Objective of the research: to carry out a theoretical and applied study of approximate solutions of the Riemann problem for two-phase immiscible fluid flow based on the Buckley – Leverett model obtained by the vanishing viscosity method.

Research objectives:

a) analysis of mathematical models of two-phase filtration processes in porous media;

b) formulation of the Riemann problem with piecewise-constant initial data having a single discontinuity, and application of the vanishing viscosity method through approximate solutions of the general Cauchy problem in solving the Riemann problem;

c) systematization of the theory of weak and entropy solutions for hyperbolic conservation laws, and justification of the uniqueness and stability of entropy solutions of the Cauchy and Riemann problems for the Buckley – Leverett equation;

d) proof of the convergence of viscous approximations to the entropy solution using the vanishing viscosity method, and analysis of the structure of the shock layer for the Riemann problem in the form of a traveling wave;

e) consideration of the inverse problem and analysis of the consistency of models, construction of an applied formulation of two-phase filtration, and validation of the obtained theoretical results by numerical models;

f) conducting computational experiments using author-developed software, and construction and analysis of numerical solutions to verify the convergence of the obtained approximate solutions.

Object of the research: Mathematical models describing the flow of two-phase immiscible fluids in porous media.

Subject of the research: Entropy solutions of the Riemann problem for the Buckley – Leverett equation, and investigation of the convergence and uniqueness of approximate solutions obtained by the vanishing viscosity method.

Scientific novelty. In the dissertation, approximate solutions of the Riemann problem for the Buckley – Leverett model obtained by the vanishing viscosity method are systematically studied. The convergence of viscous approximations to the entropy solution is proved, and the structure of the shock layer of the traveling wave type is determined. It is shown that the thickness of the transition region varies in the same order as the viscosity parameter, and the stability of the obtained results is justified. In addition, the uniqueness and stability of entropy solutions in the L^1 sense are proved, and it is shown that the obtained approximate solutions have physical meaning.

Main results submitted for defense:

a) existence, uniqueness, and L^1-stability of entropy solutions to the Riemann problem for the Buckley–Leverett equation;

b) convergence of approximate solutions to the Riemann problem for the Buckley–Leverett equation, obtained by the vanishing viscosity method, to the entropy solution, as well as the structure of the shock layer of traveling wave type;

c) an extended mathematical model of two-phase flows in porous media taking into account the dependence of viscosity on pressure;

d) numerical methods for solving direct and inverse problems of two-phase filtration for second-order partial differential equations;

e) numerical algorithms for computing approximate solutions of the Riemann problem for the Buckley–Leverett equation and their convergence.

Justification of the necessity of conducting the research. The reliability and validity of the results obtained in the dissertation are ensured by the application of modern methods of subsurface hydrodynamics, functional analysis, the theory of differential equations, and computational mathematics. Since the models under consideration are nonlinear, analytical solutions can be obtained only in limited cases. In this regard, numerical methods are widely used in the work.

The correctness and stability of the results obtained in numerical computations were verified for various initial and boundary conditions, and the consistency of the solutions with their physical meaning was analyzed. The applied algorithms and computational approaches are based on modern methods widely used in computational fluid dynamics.

The main results of the dissertation are presented in publications in international peer-reviewed scientific journals and have been discussed by independent experts. This confirms the scientific validity and reliability of the obtained results.

Theoretical and practical significance of the research. The theoretical significance of the research lies in the justification of the convergence of solutions obtained by the vanishing viscosity method to the entropy solution of the Cauchy problem in the case of the Riemann problem for hyperbolic systems, as well as in proving its uniqueness. These results contribute to the development of the theory of nonlinear conservation laws.

The practical significance of the research is determined by the possibility of applying the obtained results to modeling two-phase filtration processes in porous media. The proposed approach allows for predicting productivity in advance through numerical modeling of processes such as groundwater protection, transport of fluids through porous media, and oil displacement in the petroleum industry.

Relationship of the dissertation with other research works. The dissertation work was carried out within the framework of the research project "Predictive numerical modeling of the variability of mechanical properties of poroelastic fluid-saturated media in wave seismic fields caused by natural and anthropogenic impacts, aimed at developing reliable schemes for geophysical monitoring", implemented under the grant funding program for scientific and (or) scientific-technical projects of the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan for 2025–2027 (No. AP26196267).

Personal contribution of the author. All scientific results presented in the dissertation have been obtained by the author independently or with his direct participation. Scientific advisors and co-authors provided support in discussing the formulation of the research topic and in analyzing the obtained results.

Approbation of the work. The main results of the work were presented and discussed at the following events:

– International scientific conference "Problems of Applied Mathematics and Informatics," Aktobe Regional University named after K. Zhubanov, Aktobe, Kazakhstan, November 10–11, 2017;

– Scientific seminar "Applied Mathematics," National University of Uzbekistan named after M. Ulugbek, Tashkent, Uzbekistan (seminar leader – Doctor of Physical and Mathematical Sciences, Professor Z.H. Yuldashev), October 15, 2019;

– International scientific conference "Applied Mathematics and Information Technologies," National University of Uzbekistan named after M. Ulugbek, Tashkent, Uzbekistan, November 14–15, 2019;

– Scientific seminar "Problems of Applied Mathematics and Informatics," Aktobe Regional University named after K. Zhubanov, Department of Mathematics, Aktobe, Kazakhstan (seminar leader – Doctor of Physical and Mathematical Sciences, Professor Zh.A. Sartabanov).

Publications. A total of 11 works has been published on the topic of the dissertation, including 4 articles in peer-reviewed scientific journals indexed in the Scopus database, 2 articles in the proceedings of international conferences indexed in the Scopus database, and 5 articles in collections and journals of international conferences. In addition, within the framework of the work, the software "Analysis of the Model of Two-Phase Fluid Flow in Filtration Problems Using the Vanishing Viscosity Method" was developed, for which author's certificate No. 23424 was issued by the National Institute of Intellectual Property of the Ministry of Justice of the Republic of Kazakhstan.

The main content of the dissertation. In the first section of the dissertation, the mathematical formulation of the flow of two-phase immiscible fluids in a porous medium and the Buckley–Leverett model are considered. Multiphase fluid flow in porous media is one of the important directions of modern applied mathematics and engineering sciences. Such problems arise in many practical fields, such as oil and gas reservoir development, hydrogeology, modeling of environmental processes, and control of filtration systems. The complexity of these problems is due to the immiscible nature of fluids, the formation of interfacial boundaries, and the discontinuous structure of solutions.

Physical foundations of the two-phase filtration process. The process under consideration is the joint motion of two immiscible fluids, usually water and oil, in a porous medium. The fluids do not dissolve in each other, and a distinct interface is maintained between them. The porous medium serves as a system of channels for fluid motion, and the flow process is determined by the pressure gradient.

In engineering practice, the following standard assumptions are widely used: *the fluids are incompressible; there is no mass exchange between phases; the porous medium is not deformable; the process is isothermal; gravitational and capillary effects may be neglected under certain conditions.*

These assumptions make it possible to describe the filtration process using a relatively simple but physically meaningful mathematical model.

Mass conservation laws and Darcy's law. The description of two-phase flow is based on the law of mass conservation for each phase. In the one-dimensional case, these laws are written as follows:

$$\frac{\partial(\phi S_o)}{\partial t} + \frac{\partial u_o}{\partial x} = 0, \frac{\partial(\phi S_w)}{\partial t} + \frac{\partial u_w}{\partial x} = 0,$$

where ϕ is porosity, S_o, S_w are the saturations of oil and water, and u_o, u_w are the filtration velocities of the corresponding phases.

From a physical point of view, the saturations are related by the natural constraint:

$$S_o + S_w = 1.$$

The motion of the phases is described by Darcy's law:

$$u_\alpha = -\frac{kk_{r\alpha}}{\mu_\alpha}\frac{\partial p_\alpha}{\partial x}, \alpha \in \{o, w\},$$

where k is the absolute permeability, $k_{r\alpha}$ is the relative permeability, and μ_α is the viscosity of the phase.

Fractional flow function and the Buckley – Leverett equation. Introducing the total filtration velocity

$$u = u_o + u_w$$

and performing algebraic transformations, the fractional flow function of the water phase is defined as:

$$f_w(S_w) = \frac{\lambda_w(S_w)}{\lambda_o(S_w) + \lambda_w(S_w)}, \lambda_\alpha = \frac{k_{r\alpha}}{\mu_\alpha}.$$

The nonlinear nature of the fractional flow function is directly related to the mechanism of shock wave formation. The graph of this function and its derivative are shown below *(Figure 0.1)*.

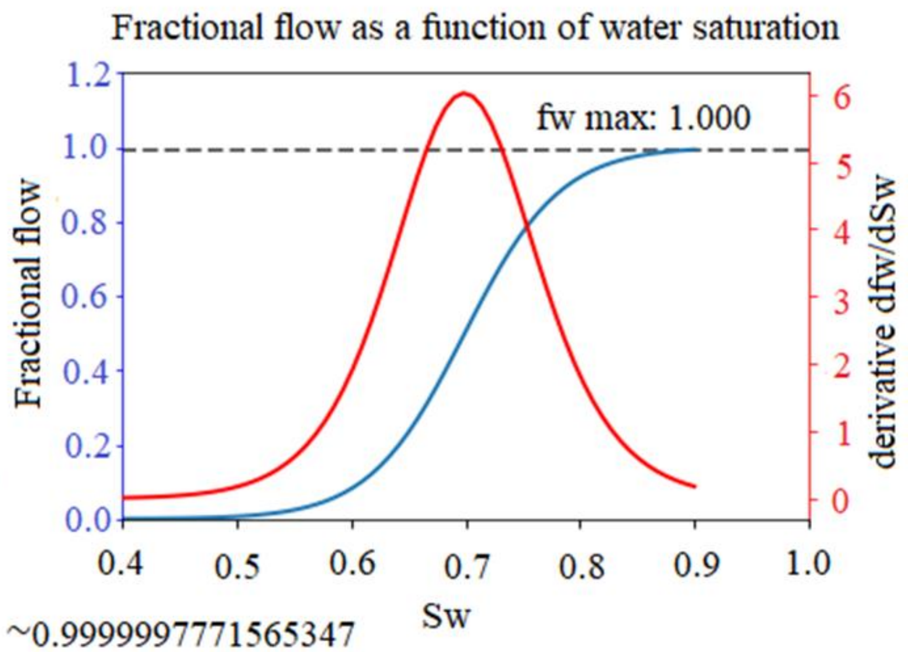

Figure 0.1 – Fractional flow function and its derivative
(obtained using the author's program)

Thus, the main evolution equation for water saturation is obtained:

$$\partial_t S_w + \partial_x f_w(S_w) = 0. \quad (0.1)$$

This equation is called the Buckley – Leverett equation and represents a hyperbolic conservation law.

Hyperbolic nature and discontinuous structure of solutions. The main feature of the Buckley–Leverett equation is that its solutions are generally discontinuous. Even if the initial condition is smooth, discontinuous changes in saturation, i.e., shock waves, may appear in finite time. From a physical point of view, this describes the formation of the displacement front. This fact shows the insufficiency of classical solutions and requires a transition to the theory of weak and subsequently entropy solutions. In the second section of the dissertation, weak and entropy solutions, the Riemann problem, and the structure of shock waves are considered.

Since the Buckley – Leverett equation is a nonlinear hyperbolic conservation law, its classical *(continuous)* solutions generally do not persist over long times. Even with smooth initial data, the gradient of the solution becomes unbounded in finite time, leading to discontinuities in saturation – shock waves. Therefore, weak and entropy formulations of the problem are introduced.

Concept of a weak solution. The absence of a classical solution requires considering the equation in the sense of distributions.

Definition 0.1 (weak solution). *If for any test function* $\varphi \in C_0^\infty(R \times [0,T))$ *the following equality holds:*

$$\int_0^T \int_R (S\ \partial_t\varphi + f(S)\ \partial_x\varphi)\ dx\ dt + \int_R S_0\ (x)\varphi(x,0)\ dx = 0,$$

then the function $S(x,t) \in L^\infty(R \times (0,T))$ *is called a weak solution of the Buckley – Leverett equation.*

This definition makes it possible to consider discontinuous solutions by transforming the differential form of the equation into an integral form.

However, weak solutions are not unique: several weak solutions may correspond to the same initial condition. Therefore, an additional condition is required to select the physically meaningful solution.

Entropy solution and Kružkov conditions. From a physical point of view, admissible solutions must not allow the increase of energy or entropy over time. This idea leads to the concept of an entropy solution.

Definition 0.2 (entropy solution). *Let* $S(x,t)$ *be a weak solution. It is called an entropy solution if for any constant* $k \in R$ *the following entropy inequality holds (in the sense of distributions):*

$$\partial_t \mid S - k \mid + \partial_x(sgn\ (S-k)(f(S) - f(k))) \le 0.$$

This condition is called the Kružkov entropy condition and ensures the selection of the physically correct direction of shock wave propagation.

For entropy solutions, the following important properties hold:

- *existence;*
- *uniqueness;*
- L^1*- stability.*

These properties ensure the applied reliability of the Buckley – Leverett model.

Riemann problem. To understand the structure of entropy solutions, the fundamental model problem – the Riemann problem is considered. Let the initial condition be given in the form:

$$S(x,0) = \begin{cases} S_L, & x < 0, \\ S_R, & x > 0, \end{cases} \quad S_L \neq S_R.$$

From a physical point of view, this problem describes a sharp change in saturation between the injection zone and the production zone.

Depending on the properties of the flux function $f(S)$, the solution may be of two types:

1. Rarefaction wave (*if* $f'(S_L) < f'(S_R)$);
2. Shock wave (*if* $f'(S_L) > f'(S_R)$).

In the Buckley – Leverett model, shock waves most often arise in practical situations.

Rankine – Hugoniot condition. The propagation speed of discontinuous solutions is determined from the integral balance:

$$\sigma = \frac{f(S_R) - f(S_L)}{S_R - S_L}.$$

This formula determines the speed of the shock front and is physically related to the conservation of mass. The geometric interpretation of this condition is explained using the chord method in the $((S, f(S))$ plane *(Figure 0.2).*

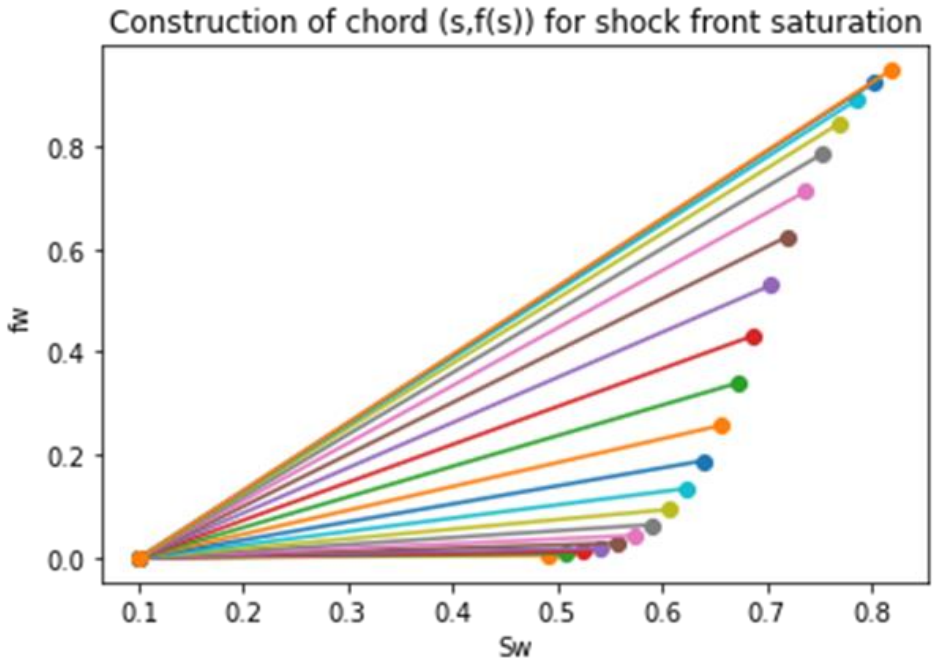

Figure 0.2 – Geometric interpretation of the Rankine – Hugoniot condition
(obtained using the author's program)

Geometric interpretation of the entropy condition. In the Buckley – Leverett model, the entropy condition can be interpreted geometrically. In the $((S, f(S))$ plane, the chord connecting the points S_L and S_R must lie below the curve (*Figure 0.1*). This interpretation clearly demonstrates the correct direction of shock propagation and the stability of the saturation front.

Vanishing viscosity method. To construct the entropy solution, the vanishing viscosity method is used. It is based on adding a small diffusion term to the hyperbolic equation:

$$\partial_t S^\varepsilon + \partial_x f(S^\varepsilon) = \varepsilon\, \partial_{xx} S^\varepsilon, \varepsilon > 0.$$

This equation is of parabolic type and is well-posed in the classical sense. The main idea is:

- for $\varepsilon > 0$ the solution is smooth;
- as $\varepsilon \to 0$ the solution converges to the entropy solution.

Main theorems

Theorem 0.1 (convergence of the vanishing viscosity limit). If $f \in C^2([0,1])$, $S_0 \in L^\infty(R)$, then for any $T > 0$ there exists a subsequence $\varepsilon_n \to 0$ such that

$$S^{\varepsilon_n} \to S, L^1_{loc}\big(R \times (0, T)\big).$$

Here, S is the entropy solution of the Buckley – Leverett equation.

Theorem 0.2 (L^1-stability). If the following inequality holds

$$\| S(\cdot, t) - \tilde{S}(\cdot, t) \|_{L^1(R)} \leq \| S_0 - \tilde{S}_0 \|_{L^1(R)}, t \geq 0,$$

then the entropy solution of the Buckley – Leverett equation (0.1) is unique.

In-depth analysis of the vanishing viscosity method and its applied meaning. To construct entropy solutions of the hyperbolic Buckley–Leverett equation, the vanishing viscosity method is used. The analytical structure of this method, the microstructure of the shock layer, and the physical interpretation of the obtained results are studied in detail.

Viscous regularized equation and its properties. By adding a small diffusion term to the hyperbolic equation, the following parabolic equation is obtained:

$$\partial_t S^\varepsilon + \partial_x f(S^\varepsilon) = \varepsilon\, \partial_{xx} S^\varepsilon, \varepsilon > 0.$$

This equation has the following properties:

- it is of parabolic type – a classical solution exists and is unique;
- the maximum principle holds:

$$\| S^\varepsilon \|_{L^\infty} \leq \| S_0 \|_{L^\infty};$$

- energy estimates ensure stability independent of ε.

These properties make it possible to study convergence as $\varepsilon \to 0$.

Structure of the shock layer as a traveling wave. For the Riemann problem, the solution of the viscous equation near the shock is sought in the special form:

$$S^{\varepsilon}(x,t) = U(\xi), \xi = \frac{x - \sigma t}{\varepsilon}.$$

The solution of the viscous problem forms a smooth transition layer near the shock. The profile of this layer depends on the parameter ε and becomes thinner as ε decreases. A typical profile of the shock layer is shown in *Figure 0.3* below.

This substitution reduces the equation to an autonomous second-order differential equation:

$$\varepsilon(-\sigma U' + f'(U)U') = \varepsilon U''.$$

After simplification:

$$U'' = (f'(U) - \sigma)U'.$$

Boundary conditions:

$$U(-\infty) = S_L, U(+\infty) = S_R.$$

This equation is analyzed in the phase plane. If $f''(S) \geq 0$ *(a convex function)*, then a monotone traveling wave profile exists.

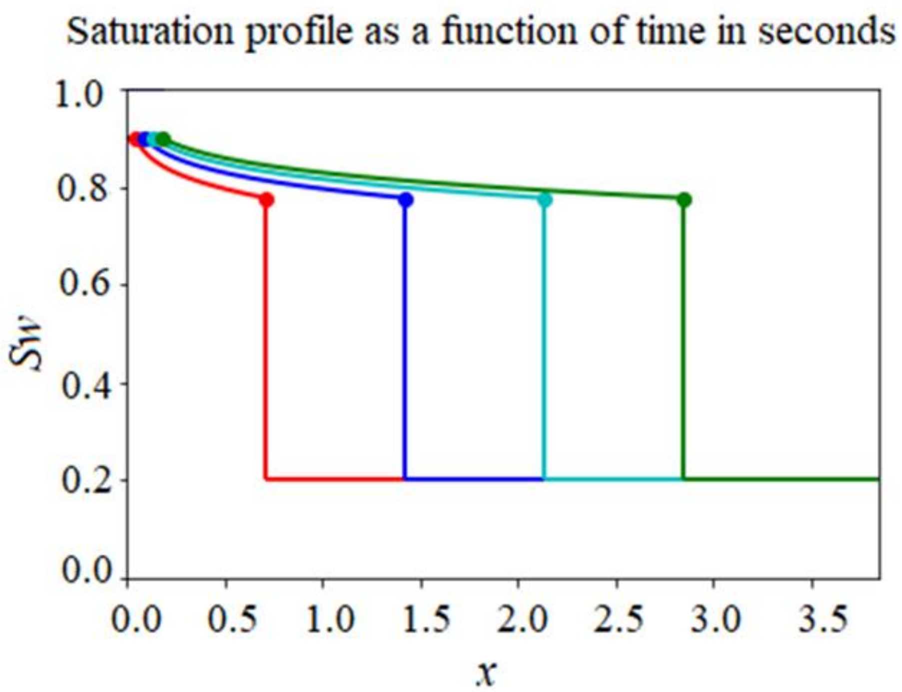

Figure 0.3 – Water saturation profile depending on time t and spatial coordinate x *(obtained using the author's program)*

Main result. The characteristic thickness of the shock layer:

$$\delta_\varepsilon = \mathcal{O}(\varepsilon).$$

From a physical point of view, this result shows that:

- as ε decreases, the shock front becomes «*thinner*»,
- as $\varepsilon \to 0$, the solution converges to a discontinuous entropy solution.

Mechanism of convergence. The proof of convergence consists of three main stages:

1. Boundedness of the approximation. L^∞ estimates are independent of ε.
2. Compactness. Using the Kolmogorov – Riesz theorems, a subsequence converging in L^1_{loc} is obtained.
3. Preservation of the entropy inequality in the limit. The viscosity term tends to zero as $\varepsilon \to 0$, while the entropy inequality is preserved.

As a result, the limit function is an entropy solution of the hyperbolic equation.

Transition to the applied filtration problem. In the one-dimensional water injection regime, the equation for water saturation is:

$$\partial_t S_w + u\,\partial_x f_w(S_w) = 0.$$

Fractional flow function:

$$f_w(S_w) = \frac{\lambda_w(S_w)}{\lambda_o(S_w) + \lambda_w(S_w)}, \lambda_\alpha = \frac{k_{r\alpha}}{\mu_\alpha}.$$

Shock front velocity:

$$v_f = \frac{u(f_w(S_{shock}) - f_w(S_i))}{S_{shock} - S_i}.$$

This formula is an applied interpretation of the Rankine – Hugoniot condition.

Physical interpretation. The vanishing viscosity method is not only a mathematical tool but also has a clear physical meaning:

- the viscosity term smooths the shock;
- the transition layer represents real physical diffusive effects;
- as $\varepsilon \to 0$, we pass to an idealized hyperbolic model.

The importance of this approach in engineering modeling lies in:

- determining the propagation speed of the front;
- predicting the water breakthrough time;
- analyzing the influence of parameters.

Numerical interpretation. If numerical schemes are constructed in accordance with the vanishing viscosity principle:

- local convergence is ensured;
- artificial oscillations are eliminated;
- sharp fronts are correctly captured.

Shock front position:

$$x_f(t) = v_f t.$$

This formula is consistent with the results of numerical modeling.

Synthesis of theory and applications. As shown in this work:

– the entropy solution was constructively determined using the vanishing viscosity method;

– the structure of the shock layer was described through traveling wave analysis;

– the reliability of the model was ensured by the L^1-stability of entropy solutions;

– the obtained results were directly applied in engineering problems of filtration processes.

In the first section of the dissertation, the basic concepts and classical results of mathematical models of two-phase flow in porous media are considered, forming the theoretical foundation.

In the second section of the dissertation, the theory of weak and entropy solutions for the Buckley – Leverett equation is systematically presented. Using the Riemann problem, the mechanism of shock wave formation is explained, and it is shown that the vanishing viscosity method is a natural mathematical tool for selecting the entropy solution.